17 December 2001

PHYSICS LETTERS A

Physics Letters A 291 (2001) 389–396

www.elsevier.com/locate/pla

Dissipation in Lie–Poisson systems and the Lorenz-84 model

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Received 2 July 2001; accepted 31 October 2001

Communicated by A.P. Fordy

Abstract

The relation between dissipation and the symplectic structure of the momentum-space is studied in so(3) Lie algebra and in 2D fluid dynamics. Three kinds of dissipative mechanisms are discussed and a general bracket formalism is introduced. A chaotic dynamical system due to Lorenz, and largely studied in low-dimensional models of geophysical fluid dynamics, is analysed in its geometric and dynamical features, by means of the formalism previously introduced. A mechanism of energy transfer for this low-order model is discussed.

PACS: 03.40.Gc; 05.45

Keywords: Casimirs; Symplectic dissipation; Lorenz attractors

1. Introduction

In a recent paper [1] it has been shown that dynamical systems on the SO(3) Lie group are particularly useful to describe some of the famous models worked out by Lorenz [2,3] and Kolmogorov [4] in the field of finite-mode hydrodynamics-type systems. In the geometric approach to fluid mechanics, started by Arnold in the sixties (for a nice review see [5]), motions of an ideal fluid in a Riemannian manifold are geodesics of a right invariant metric on the Lie group Diff(M). For two-dimensional flows, which well approximate large scale planetary fluid motions, an infinite class of enstrophy-like Casimir functions are conserved. These properties are usually lost in standard truncation algorithms and low-dimensional models and interesting 'structure preserving' finite-mode systems have been recently proposed [6,7].

In this framework, we consider the SO(3) dynamical systems as the simplest (in the scale of group complexity) 'structure preserving' models in which fundamental properties of fluid dynamics are recognizable [8]. As it will be shown for the Lorenz-84 model, a simple toy-model of general atmospheric circulation, also special dissipative processes can be formally unified via a symplectic dissipation mechanism defined in this Letter. Furthermore, the physics of the model is highlighted by the Lie algebra treatment.

2. Dissipation on the SO(3) group

Given a Lie group G and a real-valued function H (possibly time-dependent) on the dual space of its Lie algebra, called also momentum space $\mathfrak{g}^*$, in the

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0375-9601/01/$ – see front matter
PII: S0375-9601(01)00764-2

local co-ordinates x^i the Lie–Poisson equations for a dynamical system with H as Hamiltonian read as

$$\dot{x}_i = C_{ij}^k x_k \partial^j H. \tag{1}$$

The tensor C_{ij}^k represents the constants of structure of the Lie algebra $\mathfrak{g}$. For a homogeneous quadratic polynomial H, the solutions of the Lie–Poisson equations are the geodesics of G. In the particular case of $G = \mathrm{SO}(3)$, from Eq. (1) we recover the motion of a free rigid body $\dot{x}_i = C_{ij}^k x_k T^{jl} x_l$ with the Hamiltonian given by the positive-definite quadratic form

$$H = \frac{1}{2} T^{ik} x_i x_k, \tag{2}$$

where T is a symmetric matrix which physically represents the inverse of the inertia tensor. As shown in [1], we recover this structure is the Lorenz-60 model, too. The Lie algebra so(3) of antisymmetric matrices is isomorphic to R^3 by the classical cross product operator defined by the Ricci tensor ε_{ij}^k. It is straightforward to show [9] that, in this formalism, $\mathfrak{g}^*$ is endowed with a Poisson bracket structure, characterised by a Poisson skew-symmetric tensor field $J_{ij} = C_{ij}^k x_k$, known as co-symplectic form. Therefore, we can write a Lie–Poisson algebra on the functions defined on $\mathfrak{g}^*$ through the following operation:

$$\{f, g\} = J_{ij} \partial^i f \partial^j g, \tag{3}$$

for any functions $f, g \in C^\infty(\mathfrak{g}^*)$.

Casimir functions, or Casimirs, are algebraically given by the kernel of the bracket (3), i.e., $\{f, g\} = 0\ \forall g \in C^\infty(\mathfrak{g}^*)$. Physically they represent the constants of motion for the system of Hamiltonian H, i.e., $\dot{f} = \{f, H\} = 0$. Furthermore, geometrically they define a foliation of the manifold $\mathfrak{g}^*$, by the mechanism of reduction [10]. The leaves of this foliation, also called co-adjoint orbits, are symplectic manifolds on which the motion is constrained.

Casimir functions of SO(3) are given by functions of the angular momentum $Z = x^i x_i$, whose square root represents the radius of the sphere S^2 on which the trajectory lies. These co-adjoint orbits form a partition of $\mathrm{so}(3)^* \cong \mathrm{R}^3$ into spheres centred at 0 and the point 0 itself. Geometrically speaking, the energy function (2) defines an ellipsoid E to which the trajectory belongs because of energy conservation. This consideration implies that the dynamical system

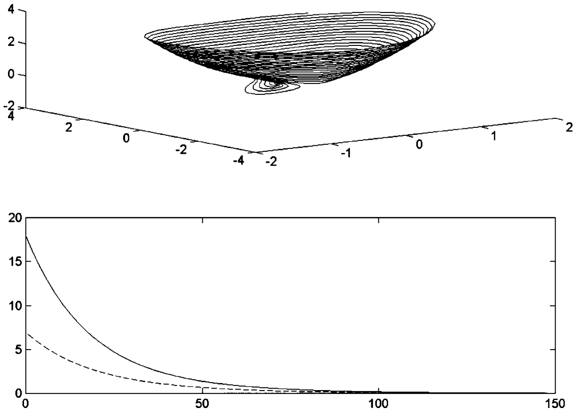

Fig. 1. Rayleigh-like dissipation in SO(3). Top: the trajectory in the phase space stops at the origin, which is an irregular point in the momentum space foliation into spheres. Bottom: evolution of Casimirs, momentum (dash), energy (solid).

evolves at the intersection of Z-Casimir and E-Casimir surfaces: $x(t) \in S^2 \cap E$.

Dissipation terms can be added to this non-canonical Hamiltonian system in two different ways: a Rayleigh-like dissipation and by a process that we call *symplectic dissipation*.

For Rayleigh-like dissipation we mean a trajectory in which the system evolves to the fixed point 0 of the momentum space. Mathematically this can be done by adding a linear term to the Hamiltonian equations (1):

$$\dot{x}_i = \{H, x_i\} + \left(D^R\right)_i^j x_j, \tag{4}$$

in which D_{ij}^R is a diagonal matrix with negative diagonal terms. When $D_{ij}^R = \lambda \delta_{ij}$, the system will evolve under a constant isotropic dissipation, otherwise it will reach the fixed point following an anisotropic evolution with respect to the principal axis x_i as in the case of Lorenz-63 model [1]. In an infinite dimensional system, the corresponding dissipation term is given by the Laplacian operator as in Navier–Stokes equations. Of course, in a Rayleigh-like dissipation, quantities as energy and Casimirs are not conserved at all. A Rayleigh-like dissipation on SO(3) is shown in Fig. 1.

The foliated geometry of the momentum space for SO(3), and in general for each Lie–Poisson system, permits dissipation processes in which some Casimirs can be preserved. This peculiar property of the system on groups has considerable applications in physics [11]. In geophysical fluid dynamics special physical processes, like cyclons merging, require a special dissipative mechanism that separates the different time scales of decay of energy and enstrophy [12]. This is a process in which the energy decays but the enstrophy remains preserved [13]. On the contrary, a mechanism of energy conservation and enstrophy dissipation is also described elsewhere [14].

In SO(3), this mechanism can be achieved by introducing the following momentum co-symplectic form

$$J^Z_{ij} = C^k_{ij}\partial_k Z, \tag{5}$$

and the energy co-symplectic form

$$J^E_{ij} = C^k_{ij}\partial_k H, \tag{6}$$

which are defined by the set of Casimirs of the Poisson algebra.

Defining the symmetric matrices

$$D^Z_{ij} = J^Z_{ik}A^{kl}J^Z_{lj} \tag{7}$$

and

$$D^E_{ij} = J^E_{ik}A^{kl}J^E_{lj}, \tag{8}$$

where A is a diagonal matrix, it is straightforward to see that by construction they have, respectively, angular momentum and energy as null eigenvectors; that is:

$$D^Z_{ij}\partial^j f(Z) = 0 \quad \text{and} \quad D^E_{ij}\partial^j f(H) = 0. \tag{9}$$

We will call operators D^Z_{ij} and D^E_{ij} as Z-preserving and E-preserving, respectively. It is then possible to define a commutative, but not Lie, algebra for each of these operators with the following brackets:

$$\langle f, g\rangle_Z = D^Z_{ij}\partial^i f\partial^j g \quad \text{and} \quad \langle f, g\rangle_E = D^E_{ij}\partial^i f\partial^j g. \tag{10}$$

For $f, g \in \mathrm{so}(3)^*$, a similar algebraic structure was introduced by Morrison in [15].

In the same way, Rayleigh-like dissipation is defined, in an algebraic form, by a similar bracket:

$$\langle f, g\rangle_R = D^R_{ij}\partial^i f\partial^j g, \tag{11}$$

where, clearly, the dissipation operator does not depend on momentum-space co-ordinates. Introducing the function:

$$\widetilde{H} = H + Z, \tag{12}$$

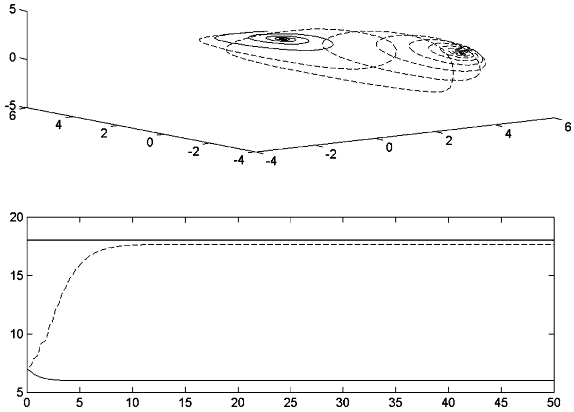

Fig. 2. Energy-preserving symplectic dissipation. Top: trajectories in the phase space, the gain (dash), or loss (solid) momentum mechanisms. Both trajectories belong to the inertia ellipsoid. Bottom: evolution for Casimirs; note the constant energy (thick solid line) and the expansion (dash) or contraction (solid) of the sphere associated with the momentum.

sum of Casimirs, that can be considered as a kind of free-energy of the system, we obtain two dissipative systems preserving Casimirs because of their algebraic structures:

$$\dot{x}_i = \{\widetilde{H}, x_i\} + \langle\widetilde{H}, x_i\rangle_Z, \quad \text{which implies} \begin{cases} \dot{H} = D^Z_{ij}\partial^i H\partial^j H, \\ \dot{Z} = 0, \end{cases} \tag{13}$$

and

$$\dot{x}_i = \{\widetilde{H}, x_i\} + \langle\widetilde{H}, x_i\rangle_E, \quad \text{which implies} \begin{cases} \dot{H} = 0, \\ \dot{Z} = D^E_{ij}\partial^i Z\partial^j Z. \end{cases} \tag{14}$$

Actually, from (13) and (14), the choice of A leads to a gain or loss of energy/momentum that is geometrically an expansion or contraction of the associated ellipsoid/sphere surface, constrained to intersect the invariant manifold. Therefore, by symplectic dissipation here we mean a mechanism for which *the motion evolves to a fixed point belonging to a Casimir surface*. Symplectic dissipative systems are shown in Figs. 2 and 3. The dissipative term of systems (13) and (14) drives the trajectory constrained to belong to the Casimir invariant manifold; therefore many dissipative behaviours can be obtained by writing a more general system:

$$\dot{x}_i = \{\widetilde{H}, x_i\} + \langle\Phi, x_i\rangle_{E,Z} \tag{15}$$

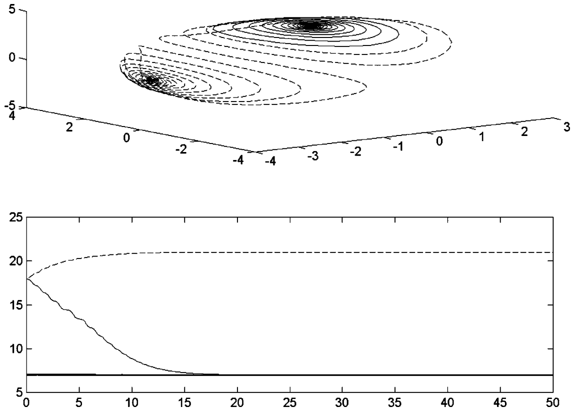

Fig. 3. Momentum-preserving symplectic dissipation. Top trajectories in the phase space. The gain (dash), or loss (solid) energy mechanisms. In this case the trajectories belong to the momentum-sphere. Bottom: evolution for Casimirs; note the constant momentum (thick solid line) and the expansion (dash) or contraction (solid) of inertia ellipsoid associated to the energy.

in which Φ gets the role of a general 'dissipative potential' on the chosen Casimir surface E or S^2.

3. Symplectic dissipation in 2D fluid dynamics

It is interesting to note that the different kinds of dissipation discussed in the last section for finite-dimensional Lie algebras can be used in the infinite-dimensional framework of fluid dynamics. We briefly show this for the barotropic equation [16] in 2D fluids.

For a stream-function $\psi(\theta_1, \theta_2)$ on a torus T^2, the dynamics is described by the Hamiltonian

$$H(\omega) = -\frac{1}{2}\int_{T^2} \psi\omega \, d\theta_1 \, d\theta_2 \tag{16}$$

and the equation of motion for the vorticity $\omega = \Delta\psi$ is given by

$$\partial_t\omega = \{H, \omega\} = \partial(\psi, \omega), \tag{17}$$

where $\partial(f, g)$ represents the 2D-Jacobian operator between two functions. The non-canonical Poisson bracket for this system is given by:

$$\{f, g\} = \int_{T^2} \omega\partial\left(\frac{\delta f}{\delta\omega}, \frac{\delta g}{\delta\omega}\right) d\theta_1 \, d\theta_2. \tag{18}$$

Casimir invariants are given by energy (17) and arbitrary functions of vorticity of the following kind:

$$C(\omega) = \int_{T^2} f(\omega) \, d\theta_1 \, d\theta_2, \tag{19}$$

like total circulation ($f(\omega) = \omega$), and enstrophy ($f(\omega) = \omega^2$) [17]; moreover, the well known 2D fluid dynamics co-symplectic operator is the Jacobian

$$J_\omega = -\partial\left(\cdot\,, \frac{\delta H}{\delta\psi}\right) = \partial(\cdot\,, \omega), \tag{20}$$

and the free energy of the system is the functional

$$\widetilde{H}(\omega) = H(\omega) + C(\omega). \tag{21}$$

As far as the dissipation terms are concerned, because of the self-adjointness of the Laplace operator [18] and the fact that $\omega = \Delta\psi$, the following symmetric bracket structure for Rayleigh-like dissipation can be introduced:

$$\begin{aligned}\langle f, g\rangle_R &= \alpha_R \int_{T^2} \frac{\delta f}{\delta\omega}\Delta^2\frac{\delta g}{\delta\omega} \, d\theta_1 \, d\theta_2 \\ &= \alpha_R \int_{T^2} \Delta\frac{\delta f}{\delta\omega}\Delta\frac{\delta g}{\delta\omega} \, d\theta_1 \, d\theta_2,\end{aligned} \tag{22}$$

where α_R is the Rayleigh dissipation coefficient. Therefore, the term $\langle H, \omega\rangle_R$ can be added to (18) giving rise to the classical Navier–Stokes equations:

$$\partial_t\omega = \{H, \omega\} + \alpha_R\langle H, \omega\rangle_R = \partial(\psi, \omega) + \alpha_R\Delta\omega, \tag{23}$$

and the well-known energy and enstrophy evolution laws

$$\begin{cases} \dot{H} = -\alpha_R C, \\ \dot{C} = -\alpha_R P, \end{cases} \tag{24}$$

where $P = \int_{T^2} (\nabla\omega)^2 \, d\theta_1 \, d\theta_2$ is the palinstrophy [19].

In the symplectic dissipation case, besides (20), we introduce another co-symplectic form,

$$J_\psi = -\partial\left(\cdot\,, \frac{\delta H}{\delta\omega}\right) = \partial(\cdot\,, \psi). \tag{25}$$

Making use of the identity for the Jacobian operator

$$\int_{T^2} f\partial(g, h) \, d\theta_1 \, d\theta_2 = -\int_{T^2} g\partial(f, h) \, d\theta_1 \, d\theta_2, \tag{26}$$

it is straightforward to proof the symmetry of the brackets

$$\langle f,g\rangle_\omega=\int_{T^2}\frac{\delta f}{\delta\omega}\partial\left(\alpha_\omega\partial\left(\frac{\delta g}{\delta\omega},\omega\right),\omega\right)d\theta_1\,d\theta_2 \\ =-\alpha_\omega\int_{T^2}\partial\left(\frac{\delta f}{\delta\omega},\omega\right)\partial\left(\frac{\delta g}{\delta\omega},\omega\right)d\theta_1\,d\theta_2 \quad (27)$$

and

$$\langle f,g\rangle_\psi=\int_{T^2}\frac{\delta f}{\delta\omega}\partial\left(\alpha_\psi\partial\left(\frac{\delta g}{\delta\omega},\psi\right),\psi\right)d\theta_1\,d\theta_2 \\ =-\alpha_\psi\int_{T^2}\partial\left(\frac{\delta f}{\delta\omega},\psi\right)\partial\left(\frac{\delta g}{\delta\omega},\psi\right)d\theta_1\,d\theta_2, \quad (28)$$

which, similarly to (7) and (8), are built by the self-adjoint operators

$$D_\omega=J_\omega\alpha_\omega J_\omega=\partial\big(\alpha_\omega\partial(\,\cdot\,,\omega),\omega\big)$$
$$\text{and}\quad D_\psi=J_\psi\alpha_\psi J_\psi=\partial\big(\alpha_\psi\partial(\,\cdot\,,\psi),\psi\big), \quad (29)$$

where α_ω and α_ψ are the symplectic dissipation coefficients (in this paper we consider only the dissipative case).

Finally, like the finite-dimensional case (13) and (14), it is possible to write a set of equations:

$$\partial_t\omega=\{\widetilde{H},\omega\}+\langle\widetilde{H},\omega\rangle_\omega \\ =\partial(\psi,\omega)+\partial\big(\alpha_\omega\partial(\psi,\omega),\omega\big) \quad (30)$$

and

$$\partial_t\omega=\{\widetilde{H},\omega\}+\langle\widetilde{H},\omega\rangle_\psi \\ =\partial(\psi,\omega)+\partial\big(\alpha_\psi\partial(\omega,\psi),\psi\big), \quad (31)$$

which preserve energy and enstrophy, respectively, while dissipating or gaining the other Casimirs:

$$\begin{cases}\dot H=\langle H,\widetilde{H}\rangle_\omega=-\alpha_\omega\langle H,H\rangle_\omega,\\ \dot C=\langle C,\widetilde{H}\rangle_\omega=0,\end{cases}$$
$$\begin{cases}\dot H=\langle H,\widetilde{H}\rangle_\psi=0,\\ \dot C=\langle C,\widetilde{H}\rangle_\psi=-\alpha_\psi\langle C,C\rangle_\psi.\end{cases} \quad (32)$$

We point out that this bracket formalism represents a comprehensive mathematical scheme including those illustrated in previous works [11–15], such that the dissipation process may act on the symplectic leaves of the chosen Casimir function.

4. The Lorenz-84 model

In 1984, E. Lorenz [20] wrote the following system of differential equations

$$\begin{cases}\dot x_1=-x_2^2-x_3^2-\alpha x_1+\alpha F_1,\\ \dot x_2=-\beta x_1x_3+x_1x_2-x_2+F_2,\\ \dot x_3=\beta x_1x_2+x_1x_3-x_3,\end{cases} \quad (33)$$

which can be regarded as a highly truncated model of the large scale atmospheric circulation (see also Ref. [21]). For certain values of parameters, Eqs. (33) show a chaotic behaviour.

This model, known in literature as Lorenz-84, was defined by its author as the '*simplest possible general circulation model*'. Although not studied as the more famous Lorenz-63, it has received a recent attention as far as the problem of evaluating the predictability of atmospheric flows is concerned [22,23]. In his original paper [20], Lorenz stated that Eqs. (33) were derived in an ad hoc manner, even if he assumed that they can be derived by a spectral truncation of some fundamental fluid dynamics equation; to the authors knowledge, it has always been considered in the literature as a conceptual model.

In Lorenz-84, the variable x_1 represents the strength of a large scale westerly zonal flow, while the others are the strengths of a chain of superposed waves (eddies) [24]. Forcing terms F_1 (zonal forcing) and F_2 (eddies forcing) are due to North–South temperature gradient and a longitudinally dependent ocean–land temperature contrast, respectively; no Coriolis term is added into the equations. A Rayleigh-like dissipation in the linear terms of (33) represents the termal and mechanical damping. The nonlinear coupling terms, containing β, represent translation of the waves by the zonal current. To our aim, it is important to know that the remaining quadratic terms of (33) represent the transfer of energy from the zonal flow to the waves (the eddies transport heat polewards and thereby reduce the North–South temperature anomaly). In what follows, we will show that this energy transfer mechanism between eddies and zonal flow can be explained by simplectic dissipation.

In a toric geometry $T^2=[0,2\pi]\times[0,\pi]$, a stream function

$$\psi(\theta_1,\theta_2)=\Gamma\big(\sqrt{2}x_1\cos\theta_2+x_2\sin\theta_2\cos\theta_1 \\ +x_3\sin\theta_1\sin\theta_2\big) \quad (34)$$

can be introduced, with x_i functions of time and a parameter Γ. The term $\psi_z = x_1 \cos\theta_2$ describes the basic zonal current, while $\psi_e = \psi - \psi_z$ describes the eddies; furthermore, a function

$$F(\theta_1,\theta_2) = F_1 \cos\theta_2 + F_2 \cos\theta_1 \sin\theta_2 \tag{35}$$

can be written in order to represent the forcing terms described above.

In terms of x_i, the mean energy (16) and the enstrophy (Eq. (19), where $f(\omega) = \omega^2$) are:

$$E = \frac{\Gamma^2}{4}\left(2x_1^2 + x_2^2 + x_3^2\right), \qquad Z = \Gamma^2\left(x_1^2 + x_2^2 + x_3^2\right), \tag{36}$$

with zonal and eddies energies

$$E_z = \frac{\Gamma^2 x_1^2}{2}, \qquad E_e = \frac{\Gamma^2(x_2^2 + x_3^2)}{4}. \tag{37}$$

For our aims, the modified barotropic equation

$$\partial_t\omega = \{H,\omega\} + \alpha_R\langle H,\omega\rangle_R + \alpha_\omega\langle H,\omega\rangle_\omega + \widetilde{F} \tag{38}$$

with H as in (16) and $\widetilde{F} = \Delta F$ will be studied in the spectral space. Choosing the coefficients $\alpha_\omega = -\frac{4}{3\Gamma^2}$ and $\beta = \frac{4\sqrt{2}\Gamma}{3\pi}$, Eq. (38) gives rise to the following dynamical system:

$$\begin{cases} \dot{x}_1 = -x_1x_2^2 - x_1x_3^2 - \alpha_R x_1 + F_1, \\ \dot{x}_2 = -\beta x_1x_3 + x_1^2x_2 - 2\alpha_R x_2 + F_2, \\ \dot{x}_3 = \beta x_1x_2 + x_1^2x_3 - 2\alpha_R x_3, \end{cases} \tag{39}$$

which is identical to the Lorenz-84 system, except for the non-linear zonal-eddies cubic interaction, different by a factor x_1. Computing the divergence of (39), it is easily verified that it can be written in terms of zonal and eddies energy terms:

$$\partial^i\dot{x}_i = 4(E_z - E_e) - 5\alpha_R. \tag{40}$$

Thus, when the zonal energy is much stronger than the eddies perturbation, the phase space volume does not contract.

Using the Young's inequality $ab \leqslant \frac{1}{2}(\varepsilon a^2 + \frac{b^2}{\varepsilon})$, $\varepsilon > 0$ [25], the system (39) can be qualitatively studied in the more physically meaningful energy space (E_z, E_e):

$$\begin{cases} \dot{E}_z = -2E_z\left(4E_e + \alpha_R - \frac{\varepsilon_1}{2}\right) + \frac{F_1^2}{2\varepsilon_1}, \\ \dot{E}_e = 4E_e\left(E_z - \alpha_R + \frac{\varepsilon_2}{16}\right) + \frac{F_2^2}{2\varepsilon_2}. \end{cases} \tag{41}$$

In the case $(\alpha_\omega < 0, \alpha_R = 0, \widetilde{F} = 0)$, Eq. (38) has an enstrophy-preserving solution and the system (41) gives rise to a 'blocked' flow where all the zonal energy is transformed into eddy energy. In the case $(\alpha_R \neq 0, \widetilde{F} = 0)$ there are no Casimirs conserved and both energy and enstrophy are dissipated. More general behaviour of (41) will be studied in a forthcoming paper; however, it is clear that the symplectic dissipation term in (38) corresponds physically to a nonlinear transfer between zonal and eddies energy added to the Hamiltonian term $\{H,\omega\}$.

In order to get the Lorenz-84 system, we write the same equation (38) with a modified enstrophy preserving dissipative potential

$$K(\omega) = -\frac{1}{2}\int_{T^2} \phi\omega \, d\theta_1\, d\theta_2, \tag{42}$$

where $\phi = \psi/x_1$. Eq. (38) reads as

$$\partial_t\omega = \{H,\omega\} + \alpha_R\langle H,\omega\rangle_R + \alpha_\omega\langle K,\omega\rangle_\omega + \widetilde{F}. \tag{43}$$

In this way, we have physically modified the interaction term between zonal and eddy flows. In the spectral space, Eq. (43) corresponds to the system:

$$\begin{cases} \dot{x}_1 = -x_2^2 - x_3^2 - \alpha_R x_1 + F_1, \\ \dot{x}_2 = -\beta x_1x_3 + x_1x_2 - 2\alpha_R x_2 + F_2, \\ \dot{x}_3 = \beta x_1x_2 + x_1x_3 - 2\alpha_R x_3, \end{cases} \tag{44}$$

which, in its quadratic terms, is equivalent to the Lorenz-84 model. In a certain range of forcing and dissipation parameters, chaos is found (Fig. 4). It is important to note that the potential $K(\omega)$ leads to a chaotic transfer between zonal and eddies energy in this model (Fig. 5).

Symplectic dissipation is then recognised to be a powerful and elegant tool to study physical processes of energy transfer in fluid dynamics by geometrical methods.

Following the algebraic approach of Section 2, we note in conclusion that the Lorenz-84 model and the equivalent system (44) can be written as the SO(3) dynamical system

$$\dot{x}_i = \left\{\widetilde{H}, x_i\right\} + \left\langle\widetilde{H}, x_i\right\rangle_R + \langle\Phi, x_i\rangle_Z + f_i, \tag{45}$$

where the Hamiltonian is that of the classical rigid body (2) added to the SO(3) Casimir; $T = diag(1, \beta + 1, \beta + 1)$ and $\Phi = \omega^k x_k$, where $\omega = (-1, 0, 0)$ is a linear Z-preserving dissipative potential. Moreover,

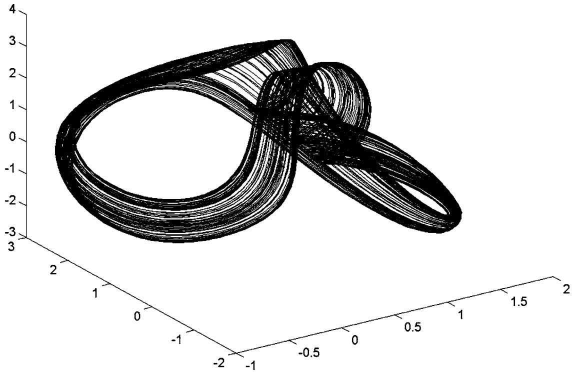

Fig. 4. Strange attractor for the system (44) ($\beta = 4,\ \alpha_R = 0.25,\ F_1 = 5,\ F_2 = 1$).

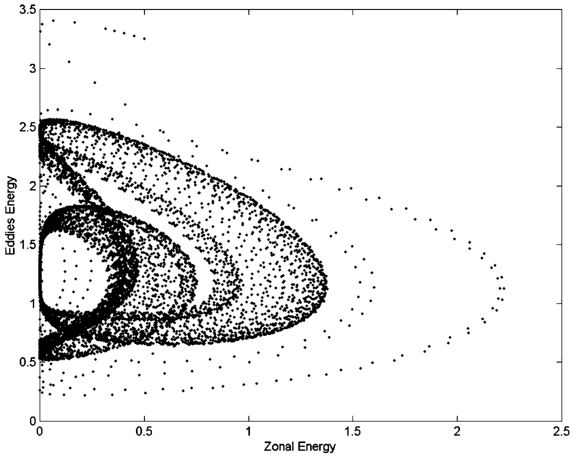

Fig. 5. Scatter plot: zonal vs. eddies energy for the modified barotropic equation (43).

defining $\Lambda = -diag(\alpha, 1, 1)$ for Lorenz-84 and $\Lambda = -diag(\alpha_R, 2\alpha_R, 2\alpha_R)$ for (44), the components of the Rayleigh-dissipation matrix are

$$D_{ij}^R = \frac{\Lambda_{ij}}{(T_{ij} + \delta_{ij})}. \tag{46}$$

It is interesting to note that the reason for the chaotic behaviour of both Lorenz-84 and (44) is soon recognised in the symplectic dissipative term. As a matter of fact, without this quadratic term, they reduce to a canonical Kolmogorov system [1].

5. Conclusions

Lie group methods for hydrodynamics-type dynamical systems are extremely useful in order to understand the complexity of fluid motions. In finite-mode approximation, conservative dynamics is described in terms of Lie–Poisson equations for a non-canonical Hamiltonian system and a number of invariants are found. As far as dissipation processes are concerned, the geometric structure of momentum space and its foliated geometry permits to define different kinds of dissipation. The first, related to the classical Rayleigh dissipation, does not conserve any Casimir function; the others, on the contrary, give rise to dissipative motions constrained to lay on an invariant manifold. The physics hidden in this geometrical scheme reveals different kinds of interaction laws between energy and other Casimirs leading to new evolution schemes for these quantities.

Applying these general laws to SO(3), the most simple object in the scale of group complexity, we obtain the somewhat intriguing result that the Lorenz-84 model is described by a combination of different Rayleigh and symplectic dissipative mechanisms. Here we recognise a chaotic transfer between zonal and eddies flows.

Finally, the bracket formalism here introduced is very general and can be applied in other fluid systems as well as in higher and most physically interesting models.

Acknowledgements

Thanks are due to Prof. V.I. Arnold for helpful correspondence concerning references and suggestions on the subject.

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