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A unified view of Kolmogorov and Lorenz systems

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Abstract

The discussion on the relation between the Kolmogorov system, considered as low-order approximation of Navier–Stokes equations, and the well-known Lorenz equations is still not completely understood. In this Letter, referring to the mathematical theory of motion on Lie groups, a particular class of Kolmogorov systems, largely studied in low-dimensional models of geophysical fluid dynamics, is extended and analysed in its geometric and dynamical features. The dynamical behaviour of this extended and unifying system generally shows chaos, contrarily to the original Kolmogorov one, and actually two well known Lorenz models, useful as toy-models in geophysical fluid dynamics, are included in it.

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1. Introduction

The failure in forecasting the dynamics of the atmospheric motion in an extended temporal range is always ascribed by the meteorological community to two principal reasons: the great number of feedback due to the non-linear physics of the system described by a set of so-called primitive equations [1], whose dynamics is driven by the Navier–Stokes equations, and the uncertainty of its initial conditions. Even for a simpler system, like a perfect fluid, these facts were established in a mile-stone theoretical article by Arnold [2], who described the Euler's equation for an incompressible fluid filled in a Riemannian manifold D, which for a planet becomes $S^2 \times [0,1]$, as a geodesic equation on the group of volume-preserving diffeomorphisms of D. The negative sectional curvature of this group implies the instability of the motion and therefore the lack of its predictability and chaos. In meteorology, this behaviour is clearly visible as in Fig. 1, where the ten-days evolution of a meteorological parameter is computed by adding small perturbations to the initial conditions, giving rise to an ensemble of possible forecasts [3].

Many attempts have been performed in order to reduce the complexity of the atmosphere to a low-di-

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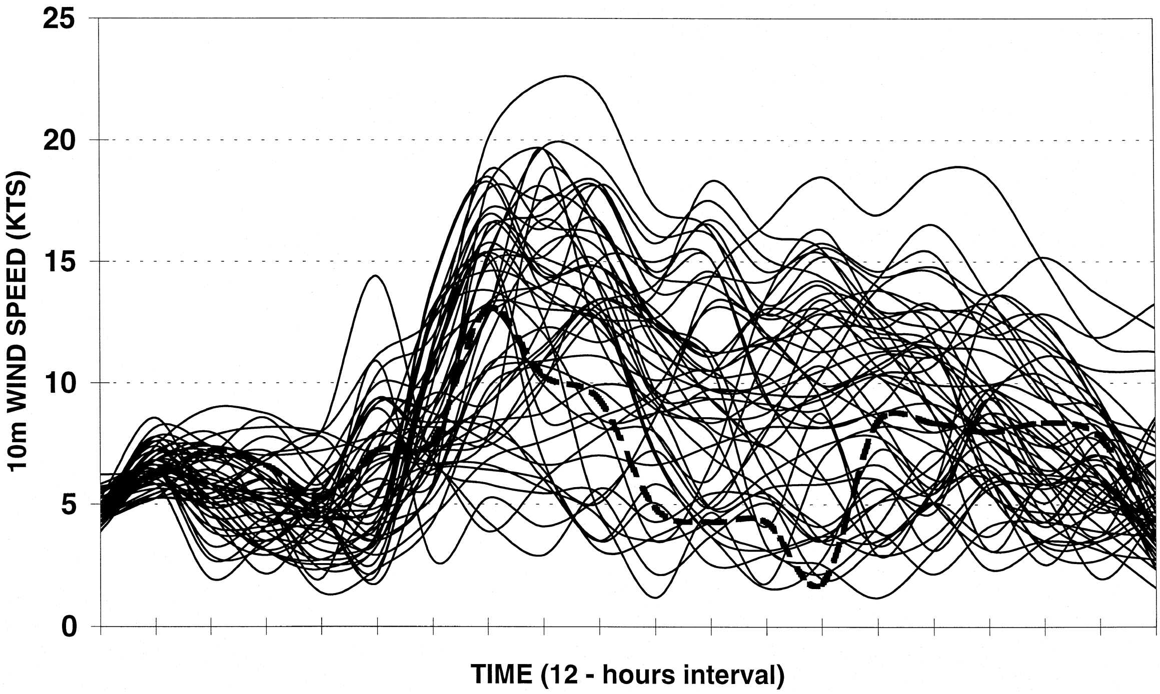

Fig. 1. 10-days prediction patterns for wind speed at 10 m above the surface on an Italian location (Brindisi), starting from slightly changes in ECMWF analysis on 13 June 1998. The dashed line represents the operative model output. (Courtesy by R. Tajani and E. Veccia on data furnished by ECMWF).

mensional dynamical system, preserving its fundamental features. The potentialities of these systems, in dynamic meteorology, were noted first by Lorenz [4], who investigated in particular a system with three degrees of freedom as a "maximum simplification" of dynamical equations for the atmosphere. Furthermore, we mention a new interesting result claiming that any dissipative equation, including Navier–Stokes, can be approximated arbitrarily by a three-dimensional system [5].

The studies on these systems have led people to search their possible chaotic properties. An interesting class of equations which represents the Galerkin finite truncation of hydrodynamic laws was introduced by A. Kolmogorov in his 1958 Moscow seminar on dynamical systems and hydrodynamic instability and recently named by V.I. Arnold as Kolmogorov system [6]. These equations reveal many similarities with the more famous Lorenz model worked out in '63 [7]. In fact, both of them describe a forced-dissipative disturbance added to a conservative dynamics. The main difference between Kolmogorov and Lorenz systems is given by their asymptotic behaviour which is stable for the former and can be chaotic for the latter.

In what follows, we will find a unifying algebraic formalism which will shed light to the difference between Lorenz and a physically meaningful class of Kolmogorov systems. Therefore, the aim of this letter is to present a main result (the formal unification through Lie algebras techniques) followed by an investigation of the physical behaviour of the system concerning its chaotic properties. Before doing this, it is necessary for our aims to introduce briefly the concept of Lie–Poisson equations on the Lie algebra of SO(3).

2. Motion on Lie groups

Motions on Lie groups are extremely interesting from the physical viewpoint and have a mathematical aesthetical appeal [8,9]. Given a group G and a real-valued function (possibly time-dependent) $h{:}T_e^*G \to \mathbb{R}$, which plays the role of a Hamiltonian, in the local co-ordinates x^i the Lie–Poisson equations for h read as

$$\dot{x}_i = C^k_{ij} x_k \partial^j h, \tag{1}$$

where the tensor C^k_{ij} represents the constants of structure of the Lie algebra $\mathfrak{g}$. It is straightforward to show [10] that, in this formalism, $\mathfrak{g}$ is endowed with a Poisson bracket structure, characterised by a Poisson skew-symmetric tensor field $\omega_{ij} = C^k_{ij} x_k$. Therefore we can write a Lie–Poisson algebra on the functions defined on $\mathfrak{g}^*$ through the following operation:

$$\{f,g\} = C^k_{ij} x_x \partial^i f \partial^j g, \tag{2}$$

for any functions $f,g \in C^\infty(\mathfrak{g}^*)$.

Casimir functions are algebraically given by the kernel of the bracket (2), i.e. $\{f,g\} = 0 \forall g \in C^\infty(\mathfrak{g}^*)$. Physically they represent the constants of motion for the system of Hamiltonian h, i.e. $\dot{f} = \{f,h\} = 0$; furthermore, geometrically they define a foliation of the manifold $\mathfrak{g}^*$, by the mechanism of reduction [11]. The leaves of this foliation, also called co-adjoint orbits, are symplectic manifolds on which the motion is constrained.

For a homogeneous quadratic polynomial h, the solutions of the Lie–Poisson equations are the geodesics of G. For a manifold M, the Lie–Poisson equation on $G = \mathrm{Diff(M)}$ represents the motion of a perfect fluid on the manifold [12]. In the particular case of $G = \mathrm{SO(3)}$, from Eq. (1) we recover the motion of a free rigid body $\dot{x}_i = C^k_{ij} x_k S^j_l x^l$ with the Hamiltonian given by the positive-definite quadratic form

$$H = \tfrac{1}{2} S^i_k x_i x^k, \tag{3}$$

where $\boldsymbol{S}$ is a symmetric matrix which physically represents the inverse of the Inertia tensor. Furthermore, in this particular case, the Lie algebra $\mathfrak{g} = \mathrm{so(3)}$ of antisymmetric matrices is isomorphic to $\mathbb{R}^3$ by the classical cross product operator defined by the Ricci tensor ε^k_{ij}, which represents the constants of structure of the group. The Casimir function of this dynamical system is given by the angular momentum $\boldsymbol{l}$, whose module

$$l^2 = x_i x^i \tag{4}$$

is computed in the standard $\mathbb{R}^3$ norm (where the co-ordinates are the components of $\boldsymbol{l}$) and whose square root represents the radius of the sphere $\mathbb{S}^2$ on which the trajectory lies. These co-adjoint orbits form a partition of $\mathrm{so(3)} \cong \mathbb{R}^3$ into spheres centred at 0 and the point 0 itself.

In the context of geophysical fluid dynamics, the following system

$$\begin{cases} \dot{x}_1 = -\left(\dfrac{1}{k^2} - \dfrac{1}{k^2+m^2}\right) km x_2 x_3, \\ \dot{x}_2 = \left(\dfrac{1}{m^2} - \dfrac{1}{k^2+m^2}\right) km x_1 x_3, \\ \dot{x}_3 = -\dfrac{1}{2}\left(\dfrac{1}{m^2} - \dfrac{1}{k^2}\right) km x_1 x_2, \end{cases} \tag{5}$$

was written by Lorenz [4] as the most simple equations of atmospheric dynamics. This low-order spectral model, derived from the Fourier transform of the barotropic vorticity equation over a torus, possesses as Casimir invariant the scalar

$$\Psi = \tfrac{1}{2}\left(x_1^2 + x_2^2 + 2x_3^2\right), \tag{6}$$

that Lorenz considered as enstrophy function (see Ref. [13] for a detailed study of the model) together with a kinetic energy

$$E = \frac{1}{4}\left(\frac{x_1^2}{k^2} + \frac{x_2^2}{m^2} + \frac{2x_3^2}{k^2+m^2}\right). \tag{7}$$

Here, k and m are specified constants depending on the size of the torus [4].

It is straightforward to show that (5) are the Lie–Poisson equations (1) for a free rigid body of Hamiltonian (7). Moreover, we note that this truncation carries into this low-order system the algebraic structures of Lie group theory used by Arnold in studying the motion of a perfect fluid. For our aim we will call this well-known system as Lorenz-60.

3. Lorenz and Kolmogorov systems

A typical set of equations which has the chaotic properties mentioned in the introduction is the famous Lorenz system [7], that we identify in the following as Lorenz-63:

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 = -x_1 x_3 + r x_1 - x_2 \,. \\ \dot{x}_3 = x_1 x_2 - \beta x_3 \end{cases} \tag{8}$$

These equations, whose geometrical properties have been recently considered by Smale [14] as one of the problems for the mathematics of the next century, were originally obtained by a drastic truncation of a system of two partial differential equations describing the Rayleigh–Bénard convection of a fluid between two parallel horizontal plates, under the assumption that the temperature difference between the upper and lower plates is kept constant. Even if they have not been considered as a proper approximation of the primitive equations for the atmosphere, they mimic very well its behaviour [15], like sensitivity to initial conditions and regimes of predictability for different regions of the attractor, which reveals its multifractal structure (Fig. 2). See also Ref. [16] for a review on the application of low-order dynamical systems in the field of weather predictions.

A system which is properly related to the Navier–Stokes equations, obtained from a 3-modes Galerkin approximation of their vorticity form, is the system

$$\dot{x}^i = A^i_{jk} x^j x^k - \Lambda^i_j x^j + f^i, \quad i = 1,2,3, \tag{9}$$

which we will call Kolmogorov system, following Arnold's paper [6]. Here Λ^i_j is a diagonal matrix. In this non-linear dynamical system the quadratic term is conservative and is added to a linear vectorial dissipation and a vectorial forcing. The third-rank tensor $\boldsymbol{A}$ has to respect the following physical constraints [17] for the unperturbed system ($\Lambda = 0$, $\boldsymbol{f} = 0$):

(i) $A^i_{jk} x_i x^j x^k = 0$ (energy conservation, where $E = \frac{1}{2} x^i x_i$);
(ii) $A^i_{ik} = 0$ (incompressibility or divergence-free condition $\partial_i \dot{x}^i = 0$).

Condition (i) can be geometrically thought as an orthogonality condition [1] between the two vectors of components x_i and $A^i_{jk} x^j x^k$. Therefore we can write

$$A^i_{jk} x^j x^k = \tfrac{1}{2}\left(\varepsilon^i_{jl} x^j B^l_k x^k + \varepsilon^i_{kl} x^k B^l_j x^j \right). \tag{10}$$

This is nothing but a convenient normalisation, since Eq. (9), in its non-linear part, can be written with the tensor A^i_{jk} replaced by its symmetrisation in the lower indexes. From the calculation of the divergence, and applying the incompressibility condition, we will obtain

$$\partial_i \dot{x}^i = \varepsilon^i_{jl} x^j B^l_i = 0; \tag{11}$$

therefore the matrix must be chosen as a symmetric one: $\boldsymbol{B} = \boldsymbol{B}^{\mathrm{T}}$.

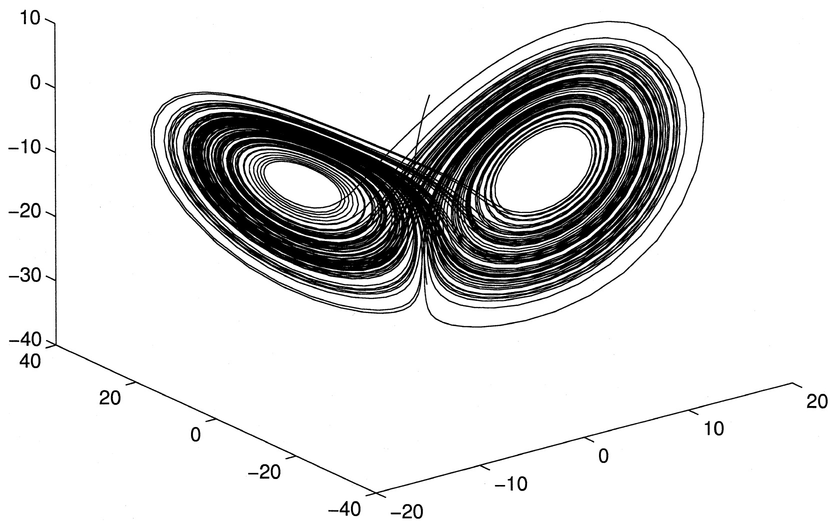

Fig. 2. A three-dimensional view of the strange attractor for Lorenz-63 (8). The parameters used are $\beta = 8/3, r = 28, \sigma = 10$.

In what follows we will refer to that large class of Kolmogorov systems where (9) can be written as:

$$\dot{x}^i = \varepsilon^i_{jk} x^k D^j_l x^l - \Lambda^i_j x^j + f^i, \quad i = 1,2,3, \tag{12}$$

where $\boldsymbol{D}$ is a diagonal matrix. As is clear from Eq. (12), we assume that our dynamical system is written in a finite-dimensional vector space whose components are the eigenbases of the friction operator.

In literature, one can find many references to the system (12), in particular in the field of dynamic meteorology [18] and geophysical fluid dynamics [19], where it is studied as a truncation of the vorticity equation. Moreover, we stress that, as shown by Obukhov [20], any inviscid hydrodynamic system in a truncated three-dimensional space is equivalent to the system of Euler equations for the dynamic of a rigid body.

In what follows, we refer to (12) as the Kolmogorov system.

4. A geometric unifying framework for Kolmogorov and Lorenz systems

Imposing the orthogonality and incompressibility conditions on the quadratic part of the system (9), we have shown that $\boldsymbol{B} = \boldsymbol{B}^{\mathrm{T}}$. In a frame where $\boldsymbol{B}$ is diagonal, the Hamiltonian component of the Kolmogorov system can be expressed as

$$A^i_{jk} = \varepsilon^i_{jl} D^l_k. \tag{13}$$

This tensor obviously satisfies the antisymmetry condition $A^i_{jk} = -A^j_{ik}$. Moreover, also the Jacobi identity $A^i_{jk} A^m_{il} + A^i_{kl} A^m_{ij} + A^i_{lj} A^m_{ik} = 0$ can be shown to hold.

Then we can consider A^i_{jk} as the structure constants tensor of a Lie algebra: from this algebraic viewpoint the incompressibility condition (ii) imposed on (1) is equivalent to the unimodularity condition of the associated Lie algebra [21]. In three dimensions, this relation leads to six non-isomorphic unimodular Lie algebras. In the Bianchi classification [22], they are related to the unimodular Lie groups via the signs of the eigenvalues of $\boldsymbol{D}$.

In this Letter, we will study the case of a Bianchi *IX* algebra, considering a positive definite *constant* diagonal matrix $\boldsymbol{D}$ as the inverse of an inertia matrix (see Ref. [18] for an analogous result). Therefore, as we have seen in the previous section, the unperturbed part of the system (12) gives rise to a rigid-body dynamics with Hamiltonian

$$H = \tfrac{1}{2} D^l_j x_l x^j \tag{14}$$

invariant under SO(3) transformations, angular momentum components x^i and body angular velocity $\Omega^i = d^i x^i$, where $(d^i)^{-1}$ are the eigenvalues of the inertia matrix (we will consider $d_1 \leqslant d_2 \leqslant d_3$). The Kolmogorov system (12) is, then, in its non-linear component, a set of Lie–Poisson equations with Hamiltonian (14).

In particular, as the geodesic motion on the Diffeomorphism group $G = \mathrm{Diff(M)}$ represents the motion of a perfect fluid on M, in this system we recover the same kind of dynamics on SO(3). This fact can be seen as a proof from the group theory viewpoint of the same “maximum simplification” results of Lorenz in his ’60 model for atmospheric motions. Moreover, a mechanical interpretation of (12) is the motion of a rigid body involving an external angular momentum and friction [12].

As far as the Lorenz equations (8) are concerned, independently from their dynamic behaviour, they are contained into a more general dynamical system which, in a tensor formalism, reads as

$$\dot{x}^i = \varepsilon^i_{jk} D^j_l x^l x^k + M^i_k x^k, \tag{15}$$

where $\boldsymbol{M}$ is a generic 3×3 real matrix. In particular, for the Lorenz system

$$\boldsymbol{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_1 - 1 & 0 \\ 0 & 0 & d_1 - 1 \end{bmatrix}, \tag{16}$$

$$\boldsymbol{M} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}. \tag{17}$$

We point out that the inertia operator D reveals a degeneracy in its eigenvalues for this peculiar system [23]. At this point, we realise that the quadratic part of both Lorenz and Kolmogorov models reveals the same algebraic structure; therefore, in order to study their similar or different dynamical behaviours, it will be useful to write both of them in a common formalism.

Under the translation

$$x^i \to x^i - k^i, \tag{18}$$

which is actually a fibre traslation over the so(3)* Lie algebra if we consider only the conservative part of (12), the Lorenz-like system (15) can be written as

$$\dot{x}^i = \varepsilon^i_{jk} D^j_l x^l x^k + \left(\tilde{M}^i_k - \Lambda^i_k\right) x^k + f^i, \tag{19}$$

where $\Lambda = -\mathrm{diag}\left(m^1_1, m^2_2, m^3_3\right)$,

$$\tilde{\boldsymbol{M}} = \begin{bmatrix} 0 & m^1_2 - \alpha_1 k_3 & m^1_3 - \alpha_1 k_2 \\ m^2_1 - \alpha_2 k_3 & 0 & m^2_3 - \alpha_2 k_1 \\ m^3_1 - \alpha_3 k_2 & m^3_2 - \alpha_3 k_1 & 0 \end{bmatrix}, \tag{20}$$

m^i_j are the elements of $\boldsymbol{M}$, $\alpha_1 = (d_2 - d_3)$, $\alpha_2 = (d_3 - d_1)$, $\alpha_3 = (d_1 - d_2)$,

$$f^i = \varepsilon^i_{jm} D^j_l k^l k^m - M^i_m k^m. \tag{21}$$

In order to recover a Lie algebra structure on the results of the generic translation (18), we impose the antisymmetry constraint on the off-diagonal terms of matrix $\tilde{\boldsymbol{M}}$, that will be written as the dual of a vector

$$\omega^i = \tfrac{1}{2} \varepsilon_{j.}^{\;ik} \tilde{M}^j_k \tag{22}$$

thus obtaining the following components of translation

$$k_1 = -\frac{m^3_2 + m^2_3}{\alpha_1}, \quad k_2 = -\frac{m^1_3 + m^3_1}{\alpha_2},$$
$$k_3 = -\frac{m^1_2 + m^2_1}{\alpha_3}. \tag{23}$$

Furthermore, this antisymmetry constraint enables us to write the linear term of (19) as a vector product of $\boldsymbol{x}$ by a constant vector field ω, such that the Lorenz-like system (15) can be written as

$$\dot{x}^i = \varepsilon^i_{jk} x^k \left(D^j_l x^l + \omega^j\right) - \Lambda^i_j x^j + f^i, \quad i = 1,2,3, \tag{24}$$

where $\Lambda^i_j = -\mathrm{diag}(m^1_1, m^2_2, m^3_3)$. We will call such equations Kolmogorov–Lorenz system.

The matrix (20) is also useful in order to obtain a *constrained* decomposition of *every* Lorenz-like system (15) with respect to the non-linear term represented by $\boldsymbol{D}$. In fact, we easily obtain the following matrix equation:

$$\tilde{\boldsymbol{M}}(\omega) + \boldsymbol{F}(\boldsymbol{k}) - \Lambda = \boldsymbol{M}, \tag{25}$$

where

$$\boldsymbol{F}(\boldsymbol{k}) = \begin{bmatrix} 0 & \alpha_1 k_3 & \alpha_1 k_2 \\ \alpha_2 k_3 & 0 & \alpha_2 k_1 \\ \alpha_3 k_2 & \alpha_3 k_1 & 0 \end{bmatrix} \tag{26}$$

represents the matrix form for the pure forcing term.

Solving (25) with respect to ω and $\boldsymbol{k}$, one obtains a clear expression of dissipative and forcing vectors, previously hidden inside the operator $\boldsymbol{M}$. We note that transformation (18) preserves the diagonal elements of $\boldsymbol{M}$. The components of the translation vector, connected with the pure forcing by (21) are given by (23), while the components of vector ω can be written as:

$$\omega_1 = -\frac{\alpha_2 m^3_2 - \alpha_3 m^2_3}{\alpha_1}, \quad \omega_2 = -\frac{\alpha_1 m^3_1 - \alpha_3 m^1_3}{\alpha_2},$$
$$\omega_3 = -\frac{\alpha_1 m^2_1 - \alpha_2 m^1_2}{\alpha_3}. \tag{27}$$

For instance, for the Lorenz equations we obtain:

$$\omega = \begin{pmatrix} 0 \\ 0 \\ -\sigma \end{pmatrix}, \quad \Lambda = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{bmatrix},$$
$$f = \begin{pmatrix} 0 \\ 0 \\ -\beta(r+\sigma) \end{pmatrix}. \tag{28}$$

We conclude our geometrical discussion, noting that the difference between Kolmogorov and Lorenz-like dynamical systems is in the vector $\omega \in$ so(3), which appears in our unifying system (24). This fact can shed a new light to the relationship between Lorenz and Kolmogorov dynamics, which is still under discussion [12].

5. Dynamical behaviour of Kolmogorov–Lorenz systems

Eqs. (24) of the Kolmogorov–Lorenz system are conceptually separable in three physically meaningful parts: the quadratic part, representing a *free* rigid body dynamics, the term $\varepsilon^i_{jk} x^k \omega^j + f^i$, representing the sum of a Coriolis-like term added to a constant forcing and, finally, a dissipative term given by $\Lambda^i_k x^k$. It is remarkable that, in the spectral models of the atmospheric equations, there appears a term like $\varepsilon^i_{jk} x^k \omega^j$ as a consequence of the planet's rotation [24]. Referring to a Kolmogorov–Lorenz system without forcing and dissipation:

$$\dot{x}^i = \varepsilon^i_{jk} x^k \left(D^j_l x^l + \omega^j \right), \quad i = 1,2,3, \tag{29}$$

we observe that these equations represent the Lie–Poisson equations on so(3) derived by adding a linear potential to the kinetic energy of the rigid body, giving rise to the following Hamiltonian function:

$$H_\omega = \tfrac{1}{2} D^i_k x_i x^k + \omega_i x^i. \tag{30}$$

The linear term $\omega_i x^i$ is studied in rigid bodies motion control and determines a spinning motion of a body under the action of an external torque produced by $\omega \in \mathrm{so}(3)$. The Hamiltonian (30) can be seen as a sort of mixture between that of a free rigid body and that of a spin system [25–27]; it has been also used in the study of gravitational Vlasov–Poisson equations [28]. The action of the potential added into the Hamiltonian (30) has the effect of stretching the trajectories, even maintaining them closed. Furthermore, they continue to belong to the sphere whose radius is given by the square root of the Casimir function $C(\boldsymbol{x}) = l^2 = x_i x^i$ (see Fig. 3).

As well known, from the group view-point the trajectories of a rigid body dynamics are determined by the intersection of two surfaces $\Sigma_X \cap \Sigma_E$ representing respectively the sphere given by the Casimir function for the so(3) algebra, which corresponds to the square of angular momentum $l^2 = x_i x^i$, and the energy ellipsoid $E_\omega = \tfrac{1}{2} D^l_j x_l x^j + \omega_l x^l$. We will study the behaviour of this system under forcing and dissipation given above.

From the computation of the divergence of the Kolmogorov–Lorenz system (24), we have

$$\partial_i \dot{x}^i = -\mathrm{Tr}\, \Lambda, \tag{31}$$

which reveals the non-conservative nature of the system under study. In particular, a positive trace for Λ is needed in order to prevent an unlimited expansion of the system volume in the phase space.

The fundamental dynamical property that a system like (24) must show, in order to have full physical significance, is a bounded nature in its kinematics behaviour. A computation of the time derivative of our angular momentum, considered as a Lyapunov function, with respect to the time gives:

$$\tfrac{1}{2} \dot{C} = -\Lambda^i_j x_i x^j + f^i x_i. \tag{32}$$

Defining $\lambda_\Lambda = \min\{\mathrm{eig}(\Lambda)\}$ as the lowest eigenvalue of the dissipation matrix, using the Schwartz

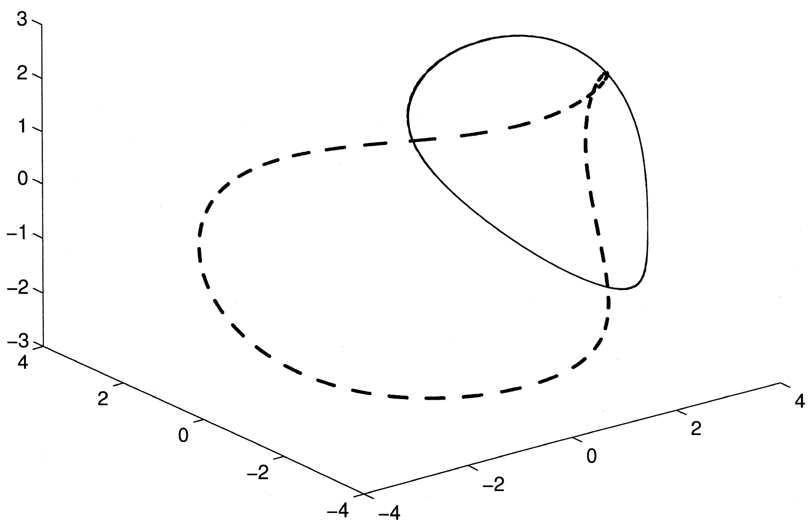

Fig. 3. Effect of the linear term $\omega_k x^k$ with $\omega = (0,3,0)$ on the dynamics of a rigid body (solid line = unperturbed rigid body motion, dashed line = motion after insertion of the linear term).

inequality and applying the Young inequality in the standard $\mathbb{R}^3$ norm $v_i v^i = ||v||^2$, i.e.

$$||a|| \cdot ||b|| \leqslant \left(\varepsilon ||a||^2 + \frac{||b||^2}{\varepsilon} \right), \quad \forall \varepsilon > 0, \tag{33}$$

we can write the following inequality:

$$\dot{C} \leqslant (-2\lambda_\Lambda + \varepsilon) C + \frac{||f||^2}{\varepsilon}. \tag{34}$$

Using Gronwall's lemma [29], we obtain the following result:

$$C \leqslant \left(C_0 + \frac{||f||^2}{\varepsilon(\varepsilon - 2\lambda_\Lambda)} \right) \mathrm{e}^{(\varepsilon - 2\lambda_\Lambda)t} + \frac{||f||^2}{\varepsilon(2\lambda_\Lambda - \varepsilon)}, \tag{35}$$

which implies bounded motion for our system as $t \to \infty$, under the hypothesis that *all* the elements of Λ are positive. In these cases, the motion of the Kolmogorov–Lorenz system (24) is confined within a sphere of radius

$$R = \frac{||f||}{\sqrt{\lambda_\Lambda}}, \tag{36}$$

which can be thought physically as the Grashoff number $\mathfrak{G}$ of the system under study; in fact, from the ratio (36) we see that the region of phase space explored by the system (24) depends on the rate of forcing and dissipation operators. It is also interesting to point out that ω has no influence in determining the size of the volume enveloping the motion.

Therefore, *a priori*, the Kolmogorov–Lorenz equations could show distinct kinds of asymptotic behaviour (limit cycle, fixed point or chaotic attractor), depending on the values of ω, on the forcing and dissipation parameters; this is relevant for establishing the predictability of this dynamical system and its complexity.

6. Steady states and stability properties

In order to study the stability of fixed points of the Kolmogorov–Lorenz system, it is known that, given a fixed-point solution of (24), one can always move it to the origin by a change of co-ordinates [30]. Therefore, it is more useful to study the stability of the origin in the system (15), where we assume that the diagonal elements of $\boldsymbol{M}$, representing dissipation, are *all* negative. Geometrically, the set of fixed points for the system (15) is given by the set $S = Q_1 \cap Q_2 \cap Q_3$ of three quadrics. Now, applying the quadric surface classification and making some algebra, we can see that the nature of Q_α depends on the degeneracy of the inertia matrix eigenvalues. In particular, for $d_1 \neq d_2 \neq d_3$ we recognise Q_α as three hyperbolic paraboloids; otherwise, for $d_1 \neq d_2 = d_3$ we obtain the plane Q_1 intersecting the two remaining hyperbolic paraboloids. The set of real solutions of the above mentioned algebraic system can be found by numerical algorithms like the pseudo-arclength continuation method [31]. However, even if we cannot give an explicit form to the bifurcation diagram of (15) in the origin, because of the many parameters involved in the linear operator $\boldsymbol{M}$, an analytic study for $\omega = 0$ can be presented.

We start by considering the Kolmogorov system (12), which, written in the "Lorenz formalism" of (15) and thanks to eqs. (25,26), assumes the form:

$$\begin{cases} x_2 x_3 - \mu_1 x_1 + k_3 x_2 + k_2 x_3 = 0, \\ x_1 x_3 - \mu_2 x_2 + k_3 x_1 + k_1 x_3 = 0, \\ x_1 x_2 - \mu_3 x_3 + k_2 x_1 + k_1 x_2 = 0, \end{cases} \tag{37}$$

where $\mu_1 = \lambda_1/\alpha_1$, $\mu_2 = \lambda_2/\alpha_2$, $\mu_3 = \lambda_3/\alpha_3$.

In this formalism it will always exist a real solution given by (0,0,0). It is easy to show that, in case of degeneracy of eigenvalues of $\boldsymbol{D}$, as in the Lorenz-63 system (8), the solution above is unique and stable. In the general case, assuming x_1 as known, we solve (37) for the remaining variables provided that $\mu_2 \mu_3 - (x_1 + k_1)^2 \neq 0$, which is always true for $\lambda_1 < \lambda_2 < \lambda_3$. Applying the Cramer's rule, we find the solutions:

$$x_2 = \frac{(k_2 x_1 + k_1 k_2 + k_3 \mu_3)}{\mu_2 \mu_3 - (x_1 + k_1)^2} x_1,$$

$$x_3 = \frac{(k_3 x_1 + k_1 k_3 + k_2 \mu_2)}{\mu_2 \mu_3 - (x_1 + k_1)^2} x_1. \tag{38}$$

Substituting (38) in the first equation of (37), we find a fifth order algebraic equation for x_1, which can always be written as $P_5(x_1) = x_1 \cdot P_4(x_1) = 0$.

Therefore we can have a maximum of five fixed points for the system (37) depending on the coefficients of the polynomial $P_4(x_1)$. Then, the set of fixed points $\mathrm{Fix}(\mu,\boldsymbol{k})$ can be decomposed as $\mathrm{Fix}(\mu,\boldsymbol{k}) = R \oplus C$, representing real and complex roots of $P_5(x_1)$; it is straightforward to prove that there are only three possible cases: $\mathrm{Fix}(\mu,\boldsymbol{k}) = 1 \oplus 4$, when all the forcing terms vanish, $Fix(\mu,\boldsymbol{k}) = 3 \oplus 2$, in case of three real solutions, and $\mathrm{Fix}(\mu,\boldsymbol{k}) = 5 \oplus 0$, when all the roots are real numbers.

Now we can study the stability of the simple solution (0,0,0) as the forcing parameters (k_1,k_2,k_3) vary. The Jacobian of (37) in (0,0,0) can be written as

$$\boldsymbol{J} = \begin{bmatrix} -\lambda_1 & \alpha_1 k_3 & \alpha_1 k_2 \\ \alpha_2 k_3 & -\lambda_2 & \alpha_2 k_1 \\ \alpha_3 k_2 & \alpha_3 k_1 & -\lambda_3 \end{bmatrix}. \tag{39}$$

First of all, (0,0,0) is stable for $k_i = 0$ $(i = 1,2,3)$, when no forcing appears in the Kolmogorov system; this case corresponds to a polynomial $P_5(x_1) = x_1 \cdot (\mu_2 \mu_3 - x_1^2)^2$, i.e. to $\mathrm{Fix}(\mu,\boldsymbol{k}) = 1 \oplus 4$. Fixing two forcings as zero and letting the third one varying, this stationary solution will reach a pitchfork bifurcation $1 \oplus 4 \to 3 \oplus 2$, *only* due to the variation of k_2 and giving rise to two stable fixed points. In fact, the eigenvalues associated to this forcing are given by

$$\sigma_{1,3} = \frac{-(\lambda_1+\lambda_3) \pm \sqrt{(\lambda_1-\lambda_3)^2 + (d_2-d_3)(d_1-d_2)k_2^2}}{2},$$

$$\sigma_2 = -\lambda_2 \tag{40}$$

and, thanks to the fact that $(d_2-d_3)(d_1-d_2) > 0$, an eigenvalue becomes positive as k_2 increases, giving rise to a coalescence of three solutions with $P_5(x_1) = x_1^3 \cdot (-2\mu_1 \mu_2 \mu_3 + x_1^2)$, which gives $\mathrm{Fix}(\mu,\boldsymbol{k}) = 3 \oplus 2$. We can conclude that, assuming only a forcing, a Kolmogorov system cannot have any kind of chaotic behaviour. We stress that the same result is obtained also in [32], where the problem is analysed studying directly the system (12). The case of more than one forcing term contemporary acting on the system can be studied by applying the Routh–Hurwitz criterion [33] on the characteristic polynomial associated to $\boldsymbol{J}$ in (0,0,0):

$$P(\sigma) = c_0 + c_1\sigma + c_2\sigma^2 + \sigma^3. \tag{41}$$

Applying this method, it results that the study of steady states can be reduced to a one-parameter bifurcation problem, depending only on k_2. Further-

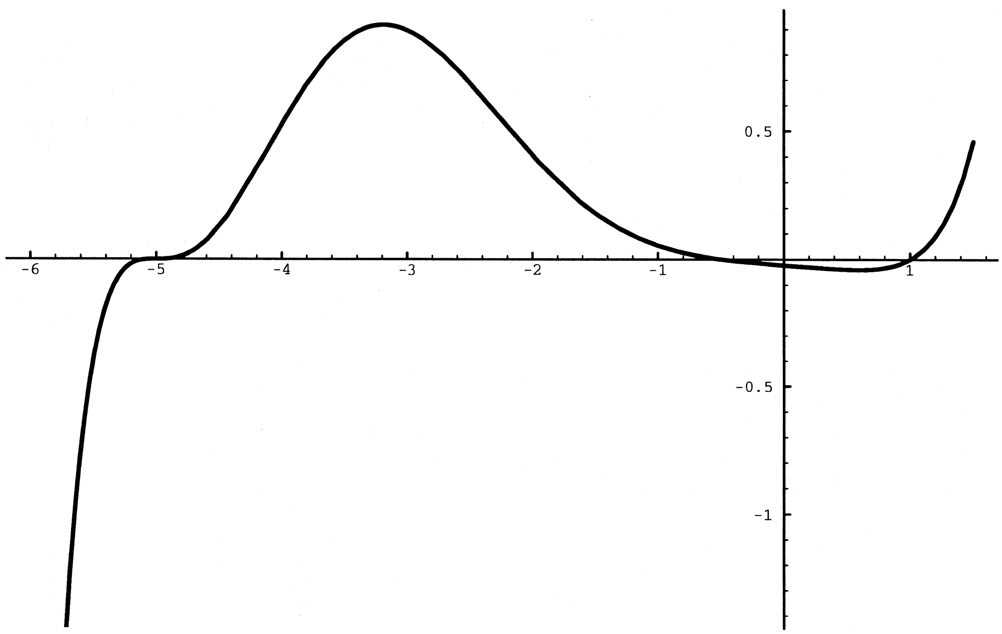

Fig. 4. Graph of the resolvent function for the study of bifurcation in the Kolmogorov system (abscissa = k_2-axis).

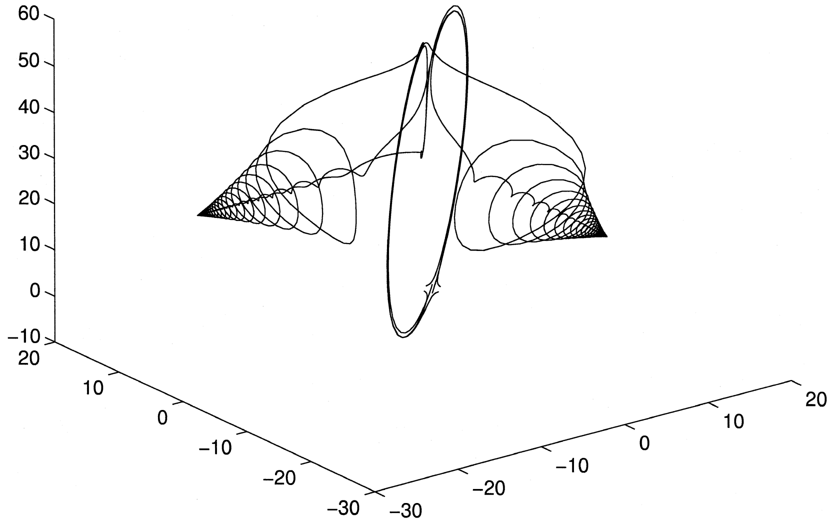

Fig. 5. Fixed points of a generic Kolmogorov system (37). Note the attractive symmetric fixed points of the dynamics.

more, the computation of the algebraic roots for the polynomial $P_4(x_1)$ by solving the associated resolvent cubic equation, leads to the graph of all the possible spectra of solutions with respect to k_2, as shown in Fig. 4.

Here we see four sectors in the k_2-axis: the boundaries of these sectors are determined by the three critical values of k_2, corresponding to coalescence of roots in $P_5(x_1) = x_1 \cdot P_4(x_1)$. From the stability point of view, the most interesting sectors are those where there is more than one real solution. In the ranges of k_2 where the function is positive, $\mathrm{Fix}(\mu,\boldsymbol{k}) = 3 \oplus 2$, we find one unstable solution (corresponding to the origin) and two stable ones. In the sector where $\mathrm{Fix}(\mu,\boldsymbol{k}) = 5 \oplus 0$ (for $k_2 < -5$ in Fig. 4), we have two stable fixed points and three unstable ones (see also Fig. 5). We note that a similar result on the number of steady states was found by Wiin-Nielsen for a low-order barotropic model of the atmosphere [34].

This analysis leads us to conclude that, even if the number of stability points depends on the forcing intensity, a Kolmogorov system in three dimensions does not show any kind of chaotic behaviour. Therefore, we can clearly see that vector ω is crucial for the birth of chaotic behaviours in these low dimensional systems.

For example, in the Lorenz-63 case, where the matrix (17) includes an *ω-term* and can be physically decomposed as in (28), we note a completely different dynamics for its eigenvalues; in fact, the couple $\mu_{1,2}$, in its motion along the real axis when the Grashoff number $\mathfrak{G}$ increases, creates an unstable linear manifold giving rise to chaos, properly due to the effect of the *ω-term* hidden in $\boldsymbol{M}$. As far as the most general set $\mathfrak{M} = \{M \in \mathrm{GL}(3,R) : \mathrm{Tr}(M) < 0\}$ is concerned, at the present stage of our study we are not yet able to give the topological features of the subset that gives rise to chaos; nevertheless, numerical experiments performed by us have always shown, in the cases of chaos, a typical bimodal Lorenz-like behaviour. The interesting result found here is that, while a pure forcing gives a bound to the volume in which the trajectories lie, the *ω-term* is responsible for the complexity of the motion.

7. Conclusions and perspectives

In 1971, Ruelle and Takens [35] proposed an alternative hypothesis to the theory of turbulence of fluids, introducing the possibility of studying such a difficult field with the aid of finite-dimensional dynamical systems and giving new impulse to the studies of classical mechanics. The Kolmogorov–Lorenz systems described above have the unifying property to collect all the possible behaviours associated to the perturbations compatible with the so(3) Lie algebra. The unperturbed geodesic motion on SO(3) has the Lorenz-60 model as geophysical counterpart in three dimensions.

We have shown that introducing a Coriolis term into the Kolmogorov structure (12) leads to the rising of chaos and to the inclusion of the Lorenz-63 system into a more general framework, which is directly derivable from a truncation of the fluid-dynamics equations. Furthermore, in our formalism, the introduction of this term derives from a natural inclusion of a linear potential $\omega_k x^k$ into the kinetic energy of the system, clarifying the structure of the non-homogeneous quadratic function associated with the energy of Lorenz attractor [12] and responsible for the chaotic behaviour of its trajectories. Moreover, the Hamiltonian structure underlying Lorenz-63 gives a natural explanation about the physical meaning of potential $\omega_k x^k$. In fact, we can see it as a contribution due to the spinning of a rigid body.

Thanks to a geometrical transformation, directly connected to the algebraic structure of the problem, this physically meaningful term, hidden inside $\boldsymbol{M}$, reveals its fundamental dynamical properties. In this optics, it is easy to prove that the classical Kolmogorov system cannot give rise to chaos in three dimensions.

In summary, we stress that this perturbation to the so(3) Lie algebra leads to extremely interesting low-order models of atmospheric motions, obtaining new results following a geometrisation program of fluid dynamics started by Arnold.

A notable remark is that this powerful Lie-group approach to hydrodynamics is fairly applied to the 3-dimensional group of rotations, obtaining a unifying dynamical system by perturbing in the right way the Lie algebraic structure of its free rigid body Hamiltonian $H = \frac{1}{2} D^l_j x_l x^j$. Obviously, the simple low-order models analysed in this letter must be considered only as toy-models of the fundamental equations of geophysical flows. Nevertheless, the underlying Lie-algebra structure of the fluid dynamic equations found in these models clarifies the source of their chaotic behaviour, as in Lorenz-63. It would be very interesting to recover the same structures in a higher spectral model coming from a bigger group like SO(N) or SU(N) [18]; we could then try to obtain some useful constraints, based on group theory, on the order of truncation of Navier–Stokes equations.

Even if in this Letter we did not classify Kolmogorov–Lorenz systems in an analytic way on the set of all possible cases, we hope that the elementary overview of such dynamical systems shown here has caught the reader's attention on the mathematical and physical richness of this subject.

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