ELSEVIER

20 April 1998

PHYSICS LETTERS A

Physics Letters A 241 (1998) 77–83

Torsion and attractors in the Kolmogorov hydrodynamical system

Antonello Pasini [1], Vinicio Pelino, Sergio Potestà

Servizio Meteorologico dell'Aeronautica, 2°CMR – Aeroporto "De Bernardi", Via di Pratica di Mare, I-00040 Pratica di Mare (Rome), Italy

Received 22 September 1997; revised manuscript received 2 February 1998; accepted for publication 4 February 1998
Communicated by A.P. Fordy

Abstract

The geometrical structure of the Kolmogorov system is studied. Considering a divergence-free geodesic motion on a Riemann–Cartan manifold, it is shown that the torsion tensor is related via group theory to the quadratic part of this system. Kolmogorov equations can be considered as the dissipative Euler–Poincaré equations on the Lie algebra of the associated group manifold. The relationship with Navier–Stokes equations and their truncated models is discussed. © 1998 Elsevier Science B.V.

PACS: 03.40.Gc; 02.40; 05.45
Keywords: Navier–Stokes equations; Kolmogorov attractors; Torsion; Teleparallelism; Euler–Poincaré equations

1. Introduction

As is well known, atmospheric motions can be described by a non-linear thermal–fluid–mechanical system, whose dynamical part is governed by the Navier–Stokes equations. Recent developments in theoretical meteorology have revealed intrinsic limits in the deterministic predictive models, leading to probabilistic approaches to long-range weather forecasting [1]. On the other hand, in 1966 Arnold [2] already recognised these limits from a geometrical point of view. In fact, the motion of a perfect fluid can be described as a dynamical system on the infinite-dimensional group of volume-preserving diffeomorphisms that has been proved to be a negative curvature manifold, therefore having divergent geodesic motions. A simplification of the Navier–Stokes equations (using, say, the Galerkin approximation) leads one to write the so-called Kolmogorov system, a set of coupled ordinary differential equations (also including the well-known Lorenz attractor) which grasp the richness of the fluid motion dynamics.

The aim of this note is to point out and study the geometrical underlying structures of this system by means of modern differential geometry techniques.

2. The Kolmogorov system

The Navier–Stokes equations for a viscous incompressible fluid in a region Ω of a differentiable manifold M can be seen as a conservative system (Euler equations) under the influence of a dissipative process and a forcing field. The configuration space for

[1] Corresponding author. E-mail: cmr2b@programatic.it.

PII S0375-9601(98)00113-3

this dynamical system is the group $G = \mathrm{Diff}_{\mathrm{vol}}(\Omega)$ of volume-preserving diffeomorphisms; therefore the velocity field takes values over its Lie algebra **g**. It is easy to recognise a Lie–Poisson structure due to the Hamiltonian structure of the Euler equations. As a matter of fact, written in the vorticity form

$$\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega - \omega \cdot \nabla \boldsymbol{u} = \nu \Delta \omega + f,$$
$$\nabla \times \boldsymbol{u} = \boldsymbol{\omega}, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where f and $\nu \Delta \omega$ represent the curl of the external forcing and the dissipative contributions, respectively, the advective (or inertial) term of the Navier–Stokes equations is the commutator of $\boldsymbol{u}$ and ω (elements of **g**). In fact, defining for $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{g}$

$$[\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a} \cdot \nabla \boldsymbol{b} - \boldsymbol{b} \cdot \nabla \boldsymbol{a}, \tag{2}$$

this operator satisfies the rules of a Lie algebra (in particular, the anticommutativity and the Jacobi identity):

$$[\boldsymbol{a}, \boldsymbol{b}] = -[\boldsymbol{b}, \boldsymbol{a}], \tag{3}$$

$$[[\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{c}] + [[\boldsymbol{b}, \boldsymbol{c}], \boldsymbol{a}] + [[\boldsymbol{c}, \boldsymbol{a}], \boldsymbol{b}] = 0. \tag{4}$$

The set **F** of functions on **g** is then endowed with a Lie–Poisson algebra defined by the so-called ideal-fluid bracket [3]

$$\{F, G\}(\boldsymbol{u}) = \int_{\Omega} \boldsymbol{u} \cdot \left[\frac{\delta F}{\delta \boldsymbol{u}}, \frac{\delta G}{\delta \boldsymbol{u}}\right] \mathrm{d}\Omega \tag{5}$$

and with the energy function of the conservative system chosen to be the kinetic energy

$$H(\boldsymbol{u}) = \int_{\Omega} ||\boldsymbol{u}||^2 \, \mathrm{d}\Omega. \tag{6}$$

The time evolution of every function F on the algebra is then given by the Poisson bracket equation [3]:

$$\dot{F} = \{F, H\}. \tag{7}$$

The annihilator of the bracket is the collection of Casimir functions, polynomials of elements of the algebra which are conserved quantities of the conservative system, like enstrophy for $\Omega \subset \mathbb{R}^2$ [4].

Then we rewrite Eq. (1) in terms of Poisson brackets,

$$\partial_t \omega + \{\boldsymbol{u}, \omega\} = \nu \Delta \omega + f,$$
$$\nabla \times \boldsymbol{u} = \omega, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{8}$$

where the bracket is defined as in (2). It is interesting to note that the above Poisson structure is also defined in geophysics fluid dynamics. In fact, adding the Coriolis term $2\Omega \times \boldsymbol{u}$ to (1), where Ω represents the Earth rotation [5], and defining the absolute vorticity

$$\omega_a = \omega + 2\Omega, \tag{9}$$

the vorticity equation will assume the following forms,

$$\partial_t \omega_a + \{\boldsymbol{u}, \omega_a\} = \nu \Delta \omega_a + f,$$
$$\nabla \times \boldsymbol{u} = \omega_a, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{10}$$

where the baroclinic vector is added to the forcing term [5].

In order to solve Eq. (1), it is useful to expand the unknowns in Fourier series using the Galerkin approximation. This leads to a system constituted by an infinite-dimensional set of coupled ordinary differential equations for the Fourier coefficients. In the field of turbulence, Kolmogorov applied this method studying the flow of an incompressible viscous fluid along a torus under the action of an external forcing and obtained the following so-called "Kolmogorov system" in Einstein notations [6]:

$$\dot{x}^i = A^i_{jk} x^j x^k - B^i_j x^j + f^i, \quad i, j, k = 1, \ldots, n, \tag{11}$$

where the first term represents the inertial term of Eq. (1), the positive definite matrix B_{ij} is the contribution of the dissipation and the last one is the external forcing; the dimensionality n of Eqs. (11) is of course the order of approximation of the Galerkin expansion. The properties of the inertial operator are given by the following physical constraints for the unperturbed system. ($B^i_j = 0$, $f^i = 0$):

(i) energy conservation, where $E = \frac{1}{2} x_i x^i$;

(ii) incompressibility (divergence-free) condition $\partial_i x_i = 0$.

Therefore it has to satisfy, respectively, two conditions [7]:

$$A^i_{jk} x_i x^j x^k = 0, \tag{12}$$

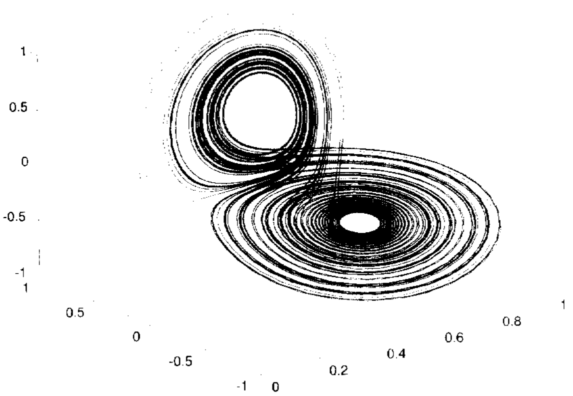

Fig. 1. A three-dimensional view of the Lorenz attractor (14), after normalization. The parameters used are $b = \frac{8}{3}$, $r = 28$, $\sigma = 10$.

$$A^i_{jk} = 0 . \tag{13}$$

It is also straightforward to demonstrate that A^i_{jk} is a third-rank tensor.

The relevance of the Kolmogorov system lies in the fact that its dynamical behaviour can be used as a toy model for the real hydrodynamical attractor of the Navier–Stokes equations. In general, we can distinguish between two different approaches in the study of hydrodynamic equations: Eulerian (in terms of velocity fields) and Lagrangian (in terms of trajectories of fluid particles). Even though these two points of view are in principle equivalent, the relationship between predictabilities, as seen in the realm of the two approaches, is still an open problem and there is no evidence of a fixed correspondence between Eulerian and Lagrangian chaotic behaviours [8]. System (11) represents the Eulerian equation for the normal modes of the velocity fields derived from the Galerkin approximation of the Navier–Stokes equations. Moreover, also the famous Lorenz attractor (Fig. 1)

$$\dot{x}_1 = -\sigma x_1 + \sigma x_2,$$
$$\dot{x}_2 = -x_1 x_3 + r x_1 - x_2,$$
$$\dot{x}_3 = x_1 x_2 - b x_3, \tag{14}$$

well known in chaos theory, and exhibiting the feature of Eulerian chaos without Lagrangian chaos [8], belongs to this class of models. This can be seen by applying the translation [7]

$$x_1 \to x_1, \quad x_2 \to x_2, \quad x_3 \to x_3 + \sigma + r, \tag{15}$$

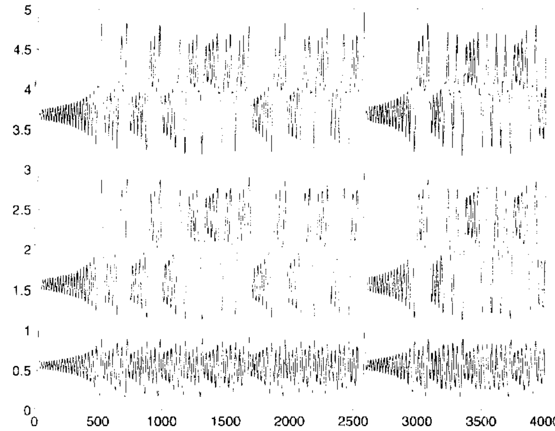

Fig. 2. Coordinates time series of the Lorenz attractor. The chaotic dynamics of the system, given by different times of permanence into each lobe of the fractal, is evident; note also the increasing of orbital radii before the transition.

which transforms (14) in a Kolmogorov system with a forcing term having vanishing components except $f_3 = -b(r + \sigma)$. In particular, this is very interesting for the studies on predictability of meteorological models; in fact, recent investigations, using the Lorenz equations, show that varying a forcing term, e.g. due to climatological effects like strong anomalies in sea-surface temperature driven by El Niño, leads to different rates of predictability in a forecasting model of tropical–extratropical interactions [9]. In Fig. 2 the coordinates time series of the Lorenz attractor (14) is shown.

For these reasons our aim is to study the system (11), and in particular the inertial operator A^i_{jk}, from a geometrical point of view.

3. Geodesic motion on Riemann–Cartan manifolds

In order to study the properties of A^i_{jk}, we apply the constraints of the previous section for a free motion of a particle on an n-dimensional differentiable manifold M with symmetric metric structure and affine connection, $M(g_{ik}, \Gamma^i_{jk})$, also known as Riemann–Cartan manifold, where in general $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

If we refer to a holonomic basis $\{e_i\}$, $i = 1, \dots, n$, and define torsion as the following $(^1_2)$ tensor T:

$$T(e_k, e_j) = T^i_{.jk} e_i = \tfrac{1}{2}(\Gamma^i_{.jk} - \Gamma^i_{.kj}) , \tag{16}$$

then it is straightforward to show that, under the most general conditions [10], a metric connection on the manifold takes the form

$$\Gamma^i_{.jk} = \overset{\circ}{\Gamma}{}^i_{.jk} + K^{.i.}_{k.j}, \tag{17}$$

where $\overset{\circ}{\Gamma}{}^i_{jk}$ is the well-known Levi–Civita connection, $K^{.i.}_{k.j}$ is the so-called contortion tensor defined in terms of torsion as

$$K^{.i.}_{k.j} = T^{.i.}_{k.j} + T^{.i.}_{k.j} + T^{i}_{.jk} \tag{18}$$

and the metricity condition

$$\nabla_i g_{jk} = 0 \tag{19}$$

holds.

The curvature tensor

$$R^{.j..}_{i.kl} = \partial_k \Gamma^j_{li} - \partial_l \Gamma^j_{ki} + \Gamma^l_{km}\Gamma^m_{li} - \Gamma^j_{lm}\Gamma^m_{ki} \tag{20}$$

can be split into the sum of the conventional Riemann tensor $\overset{\circ}{R}{}^{.j..}_{i.kl}$ (due to the contribution of the Levi–Civita connection) and of the so-called distortion tensor [11],

$$D_{ijkl} = \overset{\circ}{\nabla}_k K_{lij} - \overset{\circ}{\nabla}_l K_{kij} + K_{li.}^{..m} K_{kjm} - K_{ij}^{..m} K_{kim} \tag{21}$$

by the presence of torsion; here $\overset{\circ}{\nabla}$ is the covariant derivative whose Christoffel symbols $\overset{\circ}{\Gamma}{}^i_{jk}$ are computable from the metric tensor. It is important to stress that, because of the asymmetry of the full connection, we can have two kinds of covariant derivatives depending on the order of saturation of the indices of Γ^i_{jk} [10].

It is then possible to have manifolds with null curvature tensor even though endowed with non-vanishing Riemann and distortion tensors (this property is known as teleparallelism and its studies date back to the twenties, when attempts have been made to set up a unified theory of electromagnetism and gravity [12]). Manifolds having this property are said to be parallelized; moreover, a manifold is said to be parallelizable if its curvature tensor (20) vanishes by adding a proper amount of torsion to the connection: this is the case of group manifolds.

We point out that parallelizable manifolds are the exception rather than the rule in the set of all manifolds; for example, given a simple class of manifolds such as $M = S^n$, $n = 1, 2, \ldots$ then the only parallelizable manifolds are S^1, S^3 and S^7 which has to do with the existence of the complex numbers, the quaternions and the Cayley numbers, respectively [13].

Looking closely at $M(g_{ik}, \Gamma^i_{jk})$, it can be shown that the torsion tensor can be decomposed in the following manner with respect to the orthogonal group $SO(n)$:

$$T^{..k}_{ij.} = {}^V T^{..k}_{ij} + {}^T T^{..k}_{ij.} + {}^A T^{..k}_{ij.}, \tag{22}$$

where

$${}^V T^{..k}_{ij} = a(\delta^k_j T_i - \delta^k_i T_j), \tag{23}$$

$${}^T T^{..k}_{ij.} = b(T^{..k}_{ij.} - T^{k..}_{[ij]} - T_{[i}\delta^k_{j]}), \tag{24}$$

$${}^A T^{..k}_{ij.} = g^{kl} T_{[ijl]} \tag{25}$$

and [...] denotes antisymmetrization of the indices. The vector $T_k = T^{\cdot}_{\cdot k}$ is the Einstein–Cartan or torsion vector, result of a saturation of the torsion tensor; ${}^V T^{..k}_{ij.}$, ${}^T T^{..k}_{ij.}$, and ${}^A T^{..k}_{ij.}$ are called the "vector part", the "traceless part", and the "antisymmetric part" of torsion, respectively.

Moreover, the straightest lines, called autoparallels, coincide with the shortest ones (geodesics) if and only if the torsion is totally antisymmetric: in this particular case the contortion tensor reduces to the torsion tensor [10].

Taking as kinetic energy of the particle the positive-definite Lagrangian

$$L = \tfrac{1}{2} g_{ik} \dot{y}^i \dot{y}^k, \tag{26}$$

through the variational principle, energy conservation leads to the geodesics equation,

$$\dot{x}^i = \Gamma^i_{jk} x^j x^k, \tag{27}$$

where $x^i = \dot{y}^i$. Furthermore, the divergence-free condition gives the following equation,

$$\nabla_i \dot{x}_i = \frac{1}{\sqrt{g}} \overset{\circ}{\nabla}_k (\sqrt{g}\dot{x}^k) + T_k \dot{x}^k = 0. \tag{28}$$

In order to have the covariant constancy of the volume element $\sqrt{g}\varepsilon_{i_1} \ldots {}_{i_n}$, the geometric equivalent of a divergence-free condition, for a Riemann–Carter manifold we impose the vanishing of the Einstein–Cartan vector,

$$T_k = 0. \tag{29}$$

We choose to restrict our analysis to manifolds having a completely antisymmetric torsion tensor $T_{ijk} = {}^{A}T_{ijk}$; therefore, the motion of a free particle will be given only by $\overset{\circ}{\Gamma}{}^{i}_{kl}$ of (17).

The Ricci identities for the curvature tensor (20) in terms of torsion components read [14]

$$\sum_{(jli)} R^{.k.}_{i.jl} = \sum_{(jli)} (\nabla_j T^k_{li} - T^m_{ji} T^k_{ml}) , \tag{30}$$

where (...) denotes circular permutation of the indices. Imposing teleparallelism,

$$R_{ijkl} = 0 \tag{31}$$

and the covariant constancy of torsion,

$$\nabla_l T_{ijk} = 0 \tag{32}$$

(as a "symmetry" condition resembling the metricity condition (19)), the torsion tensor will respect the Jacobi identities

$$\sum_{(jli)} T^m_{jl} T^k_{lm} = 0 . \tag{33}$$

We can therefore conclude that a divergence-free motion on a Riemann–Cartan manifold $M(g_{ik}, \Gamma^i_{jk})$ with completely antisymmetric torsion has the same trajectories as the motion on the associated Riemann manifold $\overset{\circ}{M}(g_{ik}, \overset{\circ}{\Gamma}{}^i_{jk})$: physically this means that the trajectory of a test particle with no internal degrees of freedom will not be influenced by the torsion field. Moreover, the complete antisymmetry of torsion and the Jacobi identities (32) leads us to recognise that the components of torsion just defined have the same properties as the structure constants of a Lie algebra.

4. The geometrical structure of the inertial operator

As we have seen in Section 2, the inertial operator introduced in the Kolmogorov system derives from the truncation of the advective term of Eq. (8), which possesses the structure of Poisson brackets of the group $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$; therefore, we suppose that the tensor A^i_{jk} should conserve this algebraic property. Furthermore, we note that:

(i) because of its complete antisymmetry, T_{ijk} respects the energy conservation condition (12);

(ii) like condition (29) for the Einstein–Cartan vector, the divergence-free condition (13) implies the vanishing of the vector A_k;

(iii) the torsion tensor properties of the set of parallelized manifolds introduced before resemble a Lie algebra structure naturally endowed by these objects;

(iv) all group manifolds are parallelizable.

This leads us to consider the manifold on which the motion has been studied as a group manifold. The most important geometric property of a group manifold is that, with respect to the Levi-Civita connection, $\overset{\circ}{M}(g_{ik}, \overset{\circ}{\Gamma}{}^i_{jk})$ is a symmetric space, which means the covariant constancy of the Riemann tensor [15]

$$\overset{\circ}{\nabla}_m \overset{\circ}{R}_{ijkl} = 0 . \tag{34}$$

Therefore, a manifold of this kind is generally a curved space, but, as we have seen in the previous section, it is also parallelizable. In fact a theorem by Cartan and Schouten [16] on group manifolds states that the components of the torsion tensor to be added to the Levi-Civita connection of $\overset{\circ}{M}(g_{ik}, \overset{\circ}{\Gamma}{}^i_{jk})$ in order to transform it to the parallelized $M(g_{ik}, \Gamma^i_{jk})$ in which geodesics remain unaltered, are nothing but the group structure constants of the corresponding Lie algebra **g** expressed in vielbein terms,

$$T_{ijk} = C_{\alpha\beta\gamma} e^\alpha_i e^\beta_j e^\gamma_k . \tag{35}$$

Intuitively, this theorem is due to the fact that on a group we have a right and a left multiplication which can be associated with the two different covariant derivatives in a Riemann–Cartan space.

Conversely, the divergence-free geodesic motion on the parallelized manifold with covariantly constant torsion, Eq. (32), can be seen as the geodesic motion on the associated group manifold with a Lie algebra given by (35).

By the above correspondence we get the following system of equivalent equations [3] for the motion on $\overset{\circ}{M}(g_{ik}, \overset{\circ}{\Gamma}{}^i_{jk})$ applying the variational principle to the Lagrangian $L(x(t), x(t))$ given by (26) (Euler–Lagrange equations), and on the Lagrangian ($l(\xi(t))$) associated to the Lie algebra **g** (Euler–Poincaré equations), respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial l}{\partial \xi^\beta} - C^\gamma_{\alpha\beta}\xi^\alpha \frac{\partial l}{\partial \xi^\gamma} = 0, \qquad (36)$$

where ξ^α are the coordinates of **g**. Similarly to the rigid body, where the description of the motion can be done with respect to the space or to the body (associated with the algebra of the rotation group), these two equations describe the motion of the particle with respect to the space and to the algebra, respectively, and the positive-definite Lagrangian $l(\xi(t))$ represents the Killing form of **g**, which is then a semisimple algebra [15].

We can therefore conclude that, by identity (35), the inertial term of the Kolmogorov attractors is nothing but the structure constants of the Lie algebra of a group manifold. Moreover, the dynamics of system (11) is given by the motion on this algebra, and it is equivalent to the divergence-free motion on the associated parallelized manifold. We note that this equivalence can impose some geometrical constraints on the studies of toy-models of fluid motion like (11) and its predictability, especially in the light of new studies on the Euler–Poincaré equations [17].

As an example of the Kolmogorov attractor, we can study the motion on the Lie algebra $so(3) \simeq R^3$, for the group manifold $SO(3) \simeq S^3$. This manifold can be parallelized introducing a completely antisymmetric torsion term

$$T_{ijk} = \pm\sqrt{g}\varepsilon_{ijk} \qquad (37)$$

given by the structure constants tensor of $so(3)$. Endowing the algebra with a "rigid body" positive definite metric,

$$g_{ik} = \mathrm{diag}(I_1, I_2, I_3), \qquad (38)$$

the Lagrangian will assume the form

$$l = \tfrac{1}{2}(I_1\xi^{1^2} + I_2\xi^{3^2} + I_3\xi^{3^2}), \qquad (39)$$

and the Euler–Poincaré equations read

$$I_i\dot{\xi}^i = I_j\varepsilon^i_{jk}\xi^j\xi^k \quad (i = 1,2,3), \qquad (40)$$

which obviously are the equations for the free rigid body motion, with angular momentum $\Xi_k = I_k\xi^k$. Adding a dissipative and a forcing term, we recover the Kolmogorov equations

$$I_i\dot{\xi}^i = I_j\varepsilon^i_{jk}\xi^j\xi^k + B_{ij}\xi^j + f^i \quad (i = 1,2,3). \qquad (41)$$

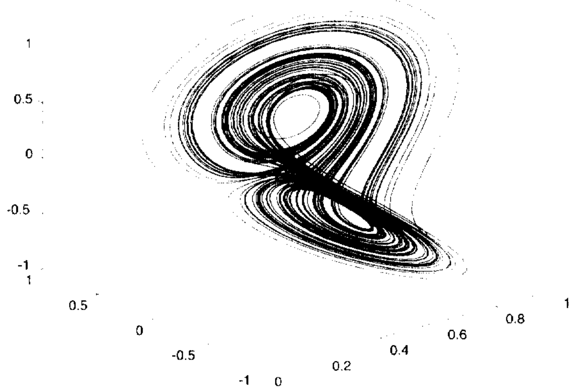

Fig. 3. Dynamics of the general $so(3)$ attractor, after normalization. The parameters used are $b = \frac{8}{3}$, $r = 28$, $\sigma = 10$, $I_1 = 3$, $I_2 = 7$, $I_3 = 4$.

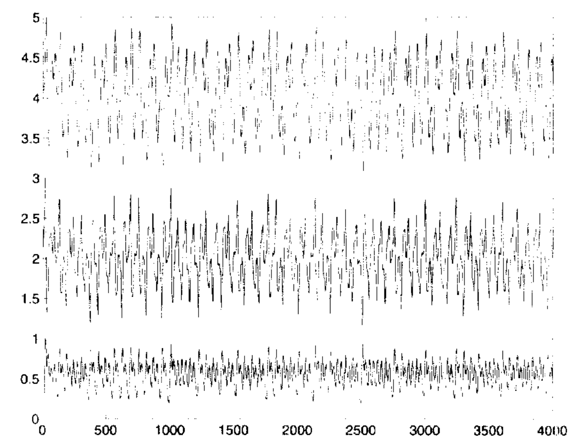

Fig. 4. Temporal variation of the $so(3)$ attractor coordinates. Note that, different from the Lorenz attractor, there is no permanence into the two lobes of the structure, but a random eight-shaped orbit with different radii.

The coefficients of the metric (38) determine the type of attractor; in fact, for $I_2 = I_3 \neq I_1$ we recover the Lorenz attractor (14), while the general case $I_1 \neq I_2 \neq I_3$ gives a similar shape (Fig. 3), but distinct dynamics as it is evident from the coordinates time series (Fig. 4).

Even though we have not presented a hydrodynamic interpretation of the attractor with distinct I_i, we would stress that it represents the general case of an ellipsoid of inertia with no symmetry in the axes, while the Lorenz attractor represents the very peculiar case of a symmetry in two of the I_i, in the realm of a more general class of the quadratic forms (39). Our conjecture (at present under investigation) is that the presence of this $SO(2)$ symmetry in Lorenz equations with respect to the Lagrangian (39) can explain the different dynamical behavior.

5. Conclusions

The conservative part of the Navier–Stokes equations exhibits interesting algebraic and geometric structures, when expressed in its vorticity form, giving rise to a Lagrangian description similar to that for the rigid body. We have tried to recover the richness of these properties of the inertial term of the Navier–Stokes equations in their truncated model known as the Kolmogorov system. Imposing the energy conservation and divergence-free constraints on the motion of a particle on a metric manifold endowed with a non-symmetric connection, we reduced to a particular class of these manifolds. As a result of this analysis, using the geometry of parallelized manifolds, we have proved that these kinds of flat manifolds can be associated with group manifolds via the relation (35), where the torsion tensor components become the structure constants of the associated Lie algebra.

In summary, this equivalence allows us to write the Kolmogorov equations, to which the Lorenz system belongs, as the Euler–Poincaré equations for the above algebra, shedding a new light on the structure of these non-linear dynamical systems. As the geodesic motion on the diffeomorphism group for a perfect fluid, we recover the same dynamics on a group in the Kolmogorov system. This means that a truncated model of the Navier–Stokes equations should conserve the algebraic properties of the Lie-Poisson structure in the inertial term, in order to be realistic. Moreover, this will imply a constraint on the dimensionality of the toy model, depending on the choice of the group considered, for instance $SO(n)$.

In perspective this work can put stronger constraints on the many toy models used to simulate atmospheric motions, and above all, using the powerful techniques of modern differential geometry, can reach a better understanding of predictability.

We are very grateful to S. Bergia for valuable suggestions.

References

[1] ECMWF, Predictability, Vols. 1,2, Seminar Proc. (1996).
[2] V.I. Arnold, Ann. Inst. Fourier (Grenoble) 16 (1966) 319.
[3] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry (Springer, New York, 1994).
[4] V.I. Arnold, B. Khesin, Ann. Rev. Fluid Mech. 24 (1992) 145.
[5] J. Pedlosky, Geophysical Fluid Dynamics, 2nd Ed. (Springer, New York, 1987).
[6] V.I. Arnold, Proc. R. Soc. (London) A 434 (1991) 19.
[7] M. Ghil, S. Childress, Topics in Geophysical Fluid Dynamics, Dynamo Theory, and Climate Dynamics (Springer, New York, 1987).
[8] A. Crisanti, M. Falcioni, G. Paladin, A. Vulpiani, Riv. Nuovo Cimento 14 (1991) 1.
[9] F. Molteni, L. Ferranti, T.N. Palmer, P. Viterbo, J. Climate 6 (1993) 777.
[10] A. Pasini, Phys. Lett. A 151 (1990) 459.
[11] B. Gogala, Int. J. Theor. Phys. 19 (1980) 573.
[12] A. Einstein, The Meaning of Relativity, 5th Ed. (Princeton Univ. Press, Princeton, 1955).
[13] C. Nash, S. Sen, Topology and Geometry for Physicists (Academic Press, London, 1983).
[14] J.A. Schouten, Ricci Calculus, 2nd Ed. (Springer, Berlin, 1954).
[15] B.A. Dubrovin, S.P. Novikov, A.T. Fomenko, Modern Geometry, Vol. 2, 2nd Ed. (Springer, New York, 1990).
[16] E. Cartan, J.A. Schouten, Proc. Kon. Ned. Akad. Amsterdam 29 (1926) 803.
[17] A. Bloch, P.S. Krisnaprasad, J.E. Marsden, T.S. Ratiu, Comm. Math. Phys. 175 (1996) 1.